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Solution of the Nonlinear Reactor

Kinetics Equations by Continuous

Analytic Continuation

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Solution of the Nonlinear Reactor
Kinetics Equations by Continuous
Analytic Continuation*

by

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requirements for the degree of Doctor of Philosophy
from The University of New Mexico.

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ABSTRACT

A method known as continuous analytic continuation is proposed for obtaining approximate solutions to the nonlinear reactor kinetics equations. The method is described and its properties are investigated theoretically. This method is one of the most powerful methods available for obtaining approximate solutions to systems of coupled nonlinear differential equations of any order and is well suited for digital computer application. Basically, the method consists of expanding those variables which are analytic functions of time in Taylor series to order K over successive intervals in the time domain.

The proposed method has several advantages over other numerical methods currently in use. The most important of these advantages is that the method yields a definitive criterion for the magnitude of the time step. This criterion is such that the time step automatically expands or contracts, depending on the behavior of the neutron level within each interval. Furthermore, the magnitude of the time step determined from this criterion can be much larger than the prompt neutron generation time. The use of this criterion to determine the time step guarantees that the error in the results increases at most linearly with the number of time steps. That is, the relative or fractional error after n time steps is bounded by $n\epsilon$ where ϵ is the error criterion ($\epsilon \ll 1$). The error criterion determines the maximum truncation error in each Taylor expansion.

Approximate solutions by continuous analytic continuation are compared with analytic solutions to the reactor kinetics equations for some of the few special cases in which analytic solutions are known. The agreement between the approximate and analytic solutions is excellent, and the error accumulation in the approximate method is, in all cases, within the limits predicted by the theory.

Comparisons are made with a numerical integration method for several cases in which analytic solutions are not available. The agreement between the two methods is good, but continuous analytic continuation is significantly faster than the numerical integration method. It is found in these problems that there is an optimum order K with respect to computing time and that the computing time is not a sensitive function of the error criterion ε. These results are in agreement with properties predicted by the theory.

Comparisons are also made with observed transients in the Godiva and SPERT I reactors following step inputs of reactivity. The calculated results using continuous analytic continuation are in good agreement with experiment in the range of reactivity inputs where the feedback model used is valid.

The reactor kinetics equations can be expressed in two different forms; one based on the prompt neutron generation time Λ and the other based on the prompt neutron lifetime ℓ. Solutions using the two forms of the equations are compared and found to be identical for all practical purposes. Since the equations based on Λ are simpler from a mathematical viewpoint than those based on ℓ, it is recommended that the equations based on Λ be used in all transient analyses.

ACKNOWLEDGMENTS

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1. INTRODUCTION

This study is devoted to the solution of the general form of the space- and energy-independent reactor kinetics equations by a method which has several important advantages over current numerical methods. Basically, the method consists of expanding those variables which are regular or analytic functions of time in Taylor series over successive intervals in the time domain. A full description of the method and an investigation of its properties are presented in Section 2.

The general form of the reactor kinetics equations, which are a system of coupled nonlinear ordinary differential equations, includes an arbitrary number of delayed neutron groups, an extraneous neutron source, and a time-varying reactivity including nonlinear power and thermal feedback.

1.1 Definition of Problem

It is not possible, in general, to obtain closed solutions to nonlinear differential equations in terms of elementary functions since at present there is no simple unifying theory in nonlinear mathematics analogous to vector spaces and operators in linear mathematics. A qualitative theory does exist[1-6] which is used to find properties of boundedness, stability, and periodicity of the solutions, but this theory does not attempt to find the solutions themselves. Thus, solutions to nonlinear differential equations must be obtained using numerical methods and other approximation techniques.

1.2 Need for a New Method

Generally, numerical methods are used to obtain approximate solutions to the nonlinear reactor kinetics equations. Among these methods are numerical integration using Simpson's rule,[7] Runge-Kutta procedures,[8,9] modified Runge-Kutta procedures,[10,11] collocation method,[12] Euler integration schemes,[13,14] finite difference methods,[15,16] and others.[17]

All of these numerical methods suffer from one or more of the following disadvantages:

    (a) Stability of the numerical procedure imposes severe limitations on the maximum permissible time step even for slow transients, thus requiring prohibitively long computing times;

    (b) There is no analytic criterion for determining the magnitude of the time step (often the procedure is one of trial and error);

    (c) Those methods which have an interval switching facility use arbitrary criteria to determine when the time step is to be modified;

    (d) The accumulated error at each time step is not known as a result of (b) and (c);

    (e) Some numerical methods are not self-starting and thus require a separate procedure to generate the first few points;

    (f) Some methods cannot handle the reactor kinetics equations in their full generality.

The proposed method, on the other hand, does not suffer from any of these disadvantages.

Analog methods are not considered in the above discussion because 'the proposed method is for use with a digital computer. Whether analog or digital methods are better at the present time is not in question here. Analog computation has its own advantages and disadvantages.[18] It is

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true that until a few years ago, analog simulation was the preferred method. Since that time, however, developments in digital computer speed and size, numerical methods, graphical display devices, and programming techniques have overcome the major objections to digital simulation.

1.3 Scope of Investigation

A digital computer program was written which solves the reactor kinetics equations using the proposed method. As a check on the accuracy of the method, comparisons were made with analytic solutions for the few special (mostly linear) cases where analytic solutions exist. Some nonlinear cases were compared with results from another numerical method. In addition, results from the proposed method were compared with some experimental excursions.

The accumulated error as a function of the number of time steps and of the truncation error criterion, and the effect of the order of the Taylor expansion and of the error criterion on the efficiency of the method were investigated theoretically. The results were then verified by comparing the approximate solutions with a few analytic cases.

There are two possible forms for the reactor kinetics equations, one based on a prompt neutron generation time, and the other on a prompt neutron lifetime. Comparisons of results for a few nonlinear cases using the two different forms of the equations were made in order to study the effect of representation.

2. DESCRIPTION OF METHOD

First, a general description of the method is given. This is followed by a description of the reactor kinetics equations and then by the application of the method to these equations. Derivations of some useful properties of the method are then made, and finally, some variations in the basic method are discussed.

2.1 General Description

The proposed method for obtaining approximate solutions to the reactor kinetics equations is a specialization to functions of a real variable of a method known as continuous analytic continuation.[19] This method is one of the most powerful available for obtaining approximate solutions to systems of coupled nonlinear differential equations of any order and is well suited to digital computer application.

In the complex plane, the method consists of expanding the dependent variables in Taylor series over successive overlapping regions along a path in the complex plane. The variables must be analytic functions within each region in which the Taylor expansion is made. For analytic functions, the convergence of the Taylor series to the value of the function is assured,[20] and a test for the convergence of the series is not required. Functions which have a finite number of singular points can be treated by this method by choosing the path from the initial point to the desired final point, so that the path neither intersects nor encloses any of the singular points.

For functions of a real variable, those dependent variables in the system of differential equations which are analytic functions are expanded in Taylor series over successive intervals on the real axis. Since for functions of a real variable the path from the initial point to the final point is confined to the real axis, the path cannot be chosen to avoid singular points. Thus, any variables which have singular points must be treated separately. This procedure is discussed in detail in Section 2.3.

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Some general features of the method are:

(a) The method is linearly iterative, that is, each successive
step is connected to the previous step by an algorithm (method
of computation) which does not increase in complexity at each
step. This is a basic requirement for any practical numerical
method.

(b) At each time step, solutions are obtained from the values of
the dependent variables and their derivatives at the previous
step only. Most other numerical methods require a knowledge
of the values of the variables at several preceding time steps
in order to extend the solution.

(c) The method yields an analytic criterion for the magnitude of
the time step at each iteration; this criterion is such that
the time step automatically expands or contracts depending
on the behavior of the function.

(d) Since the error in the approximation increases, at most,
linearly with each step, the method is more stable than some
other numerical methods.

(e) The method is self-starting; it requires only that the initial
values of the variables be specified.

These points are clarified in the sections which follow.

2.2 The Reactor Kinetics Equations

A derivation of the space-, energy-, and direction-averaged
reactor kinetics equations for systems in which fuel is stationary is
given in Appendix A. As indicated there, these equations can be expressed
in two different forms, with those based on a prompt neutron generation
time being more generally applicable. For this case, the equations to
be solved are (bars indicating effective quantities have been dropped
for convenience of notation):

$$dN(t)/dt = \Lambda^{-1}[\rho(t) - \beta]N(t) + \sum_{i=1}^{I} \lambda_i C_i(t) + S(t) \qquad (2.1)$$

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$$dC_i(t)/dt = \Lambda^{-1} \beta_i N(t) - \lambda_i C_i(t) \qquad i = 1, \ldots, I \qquad (2.2)$$

where

$N(t)$ = neutron level (which is proportional to the power level),

$C_i(t)$ = delayed neutron precursor or emitter level for type i,

$S(t)$ = extraneous neutron source,

Λ = prompt neutron generation time,

λ_i = decay constant of group i of delayed neutron emitters,

β_i = fraction of delayed neutrons in group i,

β = total delayed neutron fraction, that is

$$\beta = \sum_{i=1}^{I} \beta_i \text{ , and}$$

$\rho(t)$ = reactivity.

The reactivity $\rho(t)$ can be written in a general way as

$$\rho(t) = I(t) + F(t) \qquad (2.3)$$

where $I(t)$ is a function representing the impressed reactivity and $F(t)$ is a function which represents reactivity feedback. The impressed reactivity $I(t)$ can be in the form of an analytic function (for example, $\sin \omega t$, $e^{\alpha t}$) and/or in the form of a polynomial in t. The reactivity feedback $F(t)$ can be a function of the temperature (or temperatures of various regions) of the system, of the neutron (or power) level of the system, and of other variables, such as density, pressure, and void volume fraction. Until a specific problem is considered, the explicit form of $\rho(t)$ need not be specified.

Because of the term $\rho(t) N(t)$, Eq. 2.1 is nonlinear in $N(t)$ if $\rho(t)$ is a function of $N(t)$. For example, if the reactivity feedback $F(t)$ is

proportional to the integrated power (or total energy release) $\int N(t)\,dt$, then one of the terms in the product $\rho(t)\,N(t)$ is $N(t)\int N(t)\,dt$, which is nonlinear in $N(t)$.

Equations 2.2 for the delayed neutron emitters are linear. However, if the prompt neutron lifetime were to be used instead of the prompt neutron generation time (see Appendix A), these equations would also be nonlinear.

If thermal feedback is included in $F(t)$ and if heat transfer by radiation is present, then the heat balance equations will be nonlinear in temperature. In general, thermodynamic variables which may enter into $F(t)$ are described by nonlinear differential equations.

2.3 Application of Method to Reactor Kinetics Equations

The average neutron level and delayed neutron emitter levels, when large, are well-behaved or analytic functions of time. This is because there are no physical processes in a reactor which can result in discontinuous changes in these variables. Hence, values of these dependent variables [which are denoted in general by $y_i(t)$] at time t_{j+1} are determined from their values and derivatives at t_j by expanding in Taylor series about t_j to order K:

$$y_i(t_{j+1}) = \sum_{k=0}^{K} y_i^{(k)}(t_j)(t_{j+1} - t_j)^k/(k!) \tag{2.4}$$

In this equation, $y_i^{(k)}(t_j)$ denotes the kth derivative of $y_i(t)$ evaluated at $t = t_j$. These derivatives are obtained by successive differentiation of Eqs. 2.1-2.3.

For solution on a digital computer, it is convenient, but not necessary, to establish general expressions for the $\underline{k}$th derivative of all the dependent variables. These expressions can be established by inspection for some of the equations in the system, but others require special attention because of the occurrence of nonlinear terms. General expressions for the $\underline{k}$th derivatives of Eqs. 2.1 and 2.2 and those for several specific forms of Eq. 2.3 are given in Appendix B.

Whereas continuous analytic continuation requires information about $y_i(t_j)$, $y_i^{(1)}(t_j)$, ..., $y_i^{(k)}(t_j)$ in order to extend the solution to $y_i(t_{j+1})$, most other numerical methods require information about $y_i(t_j)$, $y_i(t_{j-1})$, ..., $y_i(t_{j-n})$ to extend the solution to $y_i(t_{j+1})$. Thus, the proposed method is self-starting and for a system of first order differential equations requires only that the initial values $y_i(0^+)$ of the variables be specified. (The derivatives at $t=0^+$ are obtained by successive differentiation of the differential equations.) However, if any of the variables $y_j(t)$ which enter into the feedback function $F(t)$ satisfy a second order differential equation, then the initial value of the first derivative $y_j^{(1)}(0^+)$ of these variables must be specified in addition; and so on for higher order equations.

The reactivity $\rho(t)$ or its derivatives may not be continuous at certain points in the transient because of impressed discontinuous changes in the function $I(t)$, such as termination of a ramp input of reactivity. Discontinuities at $t = 0$ are not a problem because the initial conditions are defined at $t = 0^+$. However, for some problems, it is possible that $\rho(t)$ cannot be expanded in a Taylor series close to certain other points in the transient. For this reason, the value of $\rho(t_{j+1})$ is obtained explicitly from its defining equation after the values of all the other dependent variables at $t = t_{j+1}$ have been obtained using Taylor expansions.

It should be noted that the algorithm (Eq. 2.4) used to compute $y_i(t_{j+1})$ does not increase in complexity at each step, but is the same for every time step. This is a basic requirement for any practical numerical method.

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2.3.1 Analytic Criterion for Time Step

The time steps $t_{j+1} - t_j$ in Eq. 2.4 are obtained by requiring that the absolute value of the relative truncation error for each expansion be at most equal to the error criterion ε (which is an input parameter $\ll 1$),

$$\left| R_i(t_{j+1})/y_i(t_{j+1}) \right| \leq \varepsilon \tag{2.5}$$

for all time steps j and variables y_i. $R_i(t_{j+1})$ is the remainder after $K + 1$ terms in the Taylor expansion of the variable y_i for the $\underline{j+1}$th time step. The remainder (in the Lagrangian form) after $K + 1$ terms in the Taylor expansion is given by[21]

$$R_i(t_{j+1}) = y_i^{(K+1)}(\eta)(t_{j+1} - t_j)^{K+1}/(K + 1)! \tag{2.6}$$

where

$$t_j \leq \eta \leq t_{j+1}$$

and

$$y_i^{(K+1)}(\eta) = \int_{t_j}^{t_{j+1}} y_i^{(K+1)}(t)dt/(t_{j+1} - t_j)$$

$$= \left[y_i^{(K)}(t_{j+1}) - y_i^{(K)}(t_j) \right]/(t_{j+1} - t_j). \tag{2.7}$$

Using Eq. 2.6, it can be seen that the largest time step which satisfies Eq. 2.5 for each variable is given by

$$(t_{j+1} - t_j)_i = \left[(K + 1)! \quad \varepsilon \left| y_i(t_{j+1}) \right| \div \left| y_i^{(K+1)}(\eta) \right| \right]^{\frac{1}{K+1}}. \tag{2.8}$$

Equation 2.8 is an analytic criterion for determining the time step. As can be seen from this equation, the time step is determined by the error

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criterion ϵ, the order K of the Taylor expansion, and by the behavior of the function $y_i(t)$ in the interval $t_j \leq t \leq t_{j+1}$. For a fixed ϵ, the time step can be increased in most problems by increasing K. For a fixed K, the time step can be increased by increasing ϵ. Also, for a fixed ϵ and K in Eq. 2.8, the time step will be largest where the ratio $\left|y_i(t_{j+1})\right| : \left|y_i^{(K+1)}(\eta)\right|$ reaches a maximum (where $y_i(t)$ is varying the least rapidly) and smallest where this ratio reaches a minimum (where $y_i(t)$ is varying the most rapidly). Thus, the time step automatically expands or contracts, depending on the behavior of the function, in order to maintain a constant relative truncation error.

Since the method requires a common time step for all the variables and since the time step computed from Eq. 2.8 is different for each variable, a common time step must be computed using the variable which yields the smallest time step. Experience with the method has shown that the neutron level N(t), which in general varies the most rapidly and over the largest range, is the variable which places the greatest restriction on the time step. Thus, the use of N(t) in Eq. 2.8 to determine a common time step will generally guarantee that the other dependent variables will also satisfy Eq. 2.5. For this reason, a common time step is determined from

$$
t_{j+1} - t_j = \left[(K + 1)! \; \epsilon \left|N(t_{j+1})\right| : \left|N^{(K+1)}(\eta)\right|\right]^{\frac{1}{K+1}} . \tag{2.9}
$$

Because $N(t_{j+1})$ and $N^{(K+1)}(\eta)$ cannot be calculated until $t_{j+1} - t_j$ is known, the time step is computed by approximating $N(t_{j+1})$ by $N(t_j)$ and $N^{(K+1)}(\eta)$ by $N^{(K+1)}(t_j)$,

$$
t_{j+1} - t_j \cong \left[(K + 1)! \; \epsilon \left|N(t_j)\right| : \left|N^{(K+1)}(t_j)\right|\right]^{\frac{1}{K+1}} . \tag{2.10}
$$

These approximations are valid for a small value of $t_{j+1} - t_j$ (obtained by using a sufficiently small ϵ), since

$$\lim_{t_{j+1} \to t_j} N^{(K+1)}(\eta) = N^{(K+1)}(t_j) \qquad \text{(see Eq. 2.7)}$$

and

$$\lim_{t_{j+1} \to t_j} N(t_{j+1}) = N(t_j) \; .$$

Regardless of the size of $t_{j+1} - t_j$, however, these approximations are valid when $N(t)$ does not change greatly in the interval $t_j \le t \le t_{j+1}$ and when the <u>K+2th</u> term in the Taylor expansion is a good approximation to the remainder after $K + 1$ terms. Furthermore, because of the $K + 1$ root in the expression for the time step, $t_{j+1} - t_j$ is not a sensitive function of the ratio $|N(t_{j+1})| \div |N^{(K+1)}(\eta)|$. For example, if $K = 4$, an order of magnitude difference between

$$|N(t_{j+1})| \div |N^{(K+1)}(\eta)|$$

and

$$|N(t_j)| \div |N^{(K+1)}(t_j)|$$

yields values of $t_{j+1} - t_j$ from Eqs. 2.9 and 2.10 which differ only by a factor of 1.6.

2.3.2 Problems in Applying Criterion and Their Solution

The use of Eq. 2.10 in conjunction with a low order K (say $K \le 2$) and large ϵ (say $\epsilon \ge 10^{-3}$) can result in relative truncation errors which are greater than ϵ over portions of some excursions. This problem arises when $N^{(K+1)}(t)$ is increasing rapidly and/or $N(t)$ is decreasing rapidly in the interval $t_{j+1} - t_j$. This problem can be eliminated by the following process:

    (a) An initial value for $t_{j+1} - t_j$ is computed from Eq. 2.10.

    (b) Using the initial value for $t_{j+1} - t_j$, initial values for

$N(t_{j+1})$ and its derivatives are computed. Because the reactor kinetics equations are coupled, this requires a pass through all the equations in the system.

(c) An improved time step is then computed from Eq. 2.9, where $N^{(K+1)}(\eta)$ is calculated from Eq. 2.7.

(d) If the improved time step is larger than its initial value (for this case the initial time step yields a smaller truncation error than required) or smaller by a factor not exceeding a preset value, the calculation proceeds to the next time step. If the improved time step does not satisfy the above conditions, it becomes the initial time step, and steps b-d are repeated until the conditions are satisfied.

Iterations on $t_{j+1} - t_j$ where j is fixed will be referred to as inner iterations, while iterations on $t_{j+1} - t_j$ where j changes will be called outer iterations. In many cases, inner iterations are not required for $\epsilon \leq 10^{-6}$, while several inner iterations per outer iteration may be required over certain portions of the transient for larger values of ϵ. Values of ϵ greater than 10^{-4} should not be used, unless accuracy is not important. For most problems, an ϵ in the range $10^{-6} - 10^{-5}$ is satisfactory. The actual value of ϵ to be used in a calculation depends on the desired accuracy which, in turn, depends on the number of outer iterations, as well as on ϵ (see Section 2.4.1).

Another problem which can arise in connection with computing the time step is that $N^{(K+1)}(t_j)$ in Eq. 2.10 may be zero. This can occur at the beginning of the transient for certain problems or at some other point in the transient for others. An example of the first case is

$$dN(t)/dt = \rho_0 \sin (\omega t) N(t)$$

where all odd derivatives of $N(t)$ at $t = 0$ are zero. An example of the second case is an inflection point in $N(t)$ when $K = 1$.

The problem just described can be handled in the following manner. If $N^{(K+1)}(t) = 0$ for $t = 0$, two choices are available: (a) the order K of the Taylor expansion can be increased or decreased until $N^{(K+1)}(0)$ is nonzero or (b) a starting time step (such as 10Λ) can be assigned and then modified, if necessary, by the process discussed previously. If $N^{(K+1)}(t)$ becomes zero during the transient ($t \neq 0$), again two choices are available: (a) the order K can be increased or decreased until $N^{(K+1)}(t)$ is nonzero or (b) the time step can be set equal to the preceding time step and then modified, if necessary, by the process described previously. In both cases, option (b) was chosen for incorporation into the digital computer program (described in Appendix D). This decision was based more on the desire to keep K constant during each run, thus allowing an investigation of the effect of K on other parameters, than on any inherent advantages of option (b) over option (a).

2.4 Derivation of Some Properties of the Method

By placing a restriction (Eq. 2.5) on the maximum value of the relative truncation error in the Taylor expansion for each variable and each time step, it was shown in Section 2.3 that a definitive criterion (Eq. 2.9) for the magnitude of each time step could be obtained. The use of this criterion to compute the time step makes it possible to obtain a first approximation to the upper limit of the accumulated relative (or fractional) error and to determine the effects of the order K of the Taylor expansion and of the error criterion ϵ on the efficiency of the method.

2.4.1. Accumulated Error

In order to simplify the notation in the following discussion, the dependent variables in the set of coupled nonlinear differential equations which describe the behavior of the reactor system are denoted by $y_i(t)$ and the set of equations by

$$dy_i(t)/dt = f_i(y_1, \ldots, y_m, t) \qquad i = 1, 2, \ldots, m. \qquad (2.11)$$

Approximate values for $y_i(t)$ from the Taylor expansions are denoted by $Y_i(t)$. Superscripts enclosed in parentheses denote the order of the derivatives.

It should be pointed out again that the method of continuous analytic continuation is not restricted to first order equations, such as Eqs. 2.11. The only restriction on order is that no derivatives occurring on the right-hand side of the differential equations can be of greater order than the derivative on the left-hand side.

The accumulated relative (or fractional) error E_j^i for variable y_i after j time steps (or after j outer iterations) is, by definition,

$$E_j^i = [y_i(t_j) - Y_i(t_j)]/Y_i(t_j) \qquad (2.12)$$

where $y_i(t_j)$ is the exact value at $t = t_j$.

It is now assumed that the coupled set of Eqs. 2.11 has been solved for some particular problem in the interval $0 \leq t \leq t_{max}$ using continuous analytic continuation and that the number of time steps at $t = t_{max}$ is n_{max}. It will be shown by mathematical induction* that a first approximation to the absolute value of the accumulated relative error after n ($1 \leq n \leq n_{max}$) time steps is given by

$$|E_n^i| \leq n \, \varepsilon. \qquad (2.13)$$

---

* If a theorem concerning positive integers n is known to be true for n = 1 and if the assumed truth of the theorem for n = j implies its truth for n = j + 1, then the theorem is true for all positive integers n.

At the initial point $t = t_o$, the values $y_i(t_o)$ of all the dependent variables are known exactly from the initial conditions. Furthermore, all the derivatives $y_i^{(k)}(t_o)$, $k = 1, 2, \ldots, K$, can be computed exactly by successive differentiation of Eqs. 2.11. Hence, for $t = t_o$, the following equality holds

$$Y_i^{(k)}(t_o) = y_i^{(k)}(t_o) \qquad \begin{array}{l} k = 0,1,\ldots K \\ i = 1,2,\ldots,m \end{array} \qquad (2.14)$$

where K is the order of the Taylor expansion and m is the number of dependent variables.

The exact values of the dependent variables at $t = t_1 (t_1 > t_o)$ are given by

$$y_i(t_1) = \sum_{k=0}^{K} y_i^{(k)}(t_o)(t_1 - t_o)^k/(k!) + R_i(t_1) \qquad (2.15)$$

where $R_i(t_1)$ is the remainder after $K + 1$ terms in the Taylor expansion for the first time step. The approximate values at $t = t_1$ are calculated from

$$Y_i(t_1) = \sum_{k=0}^{K} Y_i^{(k)}(t_o)(t_1 - t_o)^k/(k!) \qquad (2.16)$$

The combination of Eqs. 2.14 through 2.16 gives the result

$$y_i(t_1) = Y_i(t_1) + R_i(t_1). \qquad (2.17)$$

Now the time steps $t_j - t_{j-1}$ are chosen so that (Eq. 2.5)

$$\left| \frac{R_i(t_j)}{Y_i(t_j)} \right| \leq \varepsilon \qquad \begin{array}{l} i = 1,2,\ldots,m \\ j = 1,2,\ldots,n_{max} \end{array} \qquad (2.18)$$

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is satisfied for a specified $\varepsilon \ll 1$. Equation 2.17 can be rearranged to yield

$$[y_i(t_1) - Y_i(t_1)]/Y_i(t_1) = R_i(t_1)/Y_i(t_1). \tag{2.19}$$

With the definition of E_n^i from Eq. 2.12, there results on combining Eqs. 2.19 and 2.18,

$$\left| E_1^i \right| \leq \varepsilon \tag{2.20}$$

which proves that Eq. 2.13 is true for $n = 1$.

Assume that Eq. 2.13 is true for $n = j$, i.e., that

$$\left| E_j^i \right| \leq j\varepsilon \tag{2.21}$$

is a true statement. It shall be shown that this assumption implies that Eq. 2.13 is true for $n = j + 1$.

From Eq. 2.12, $y_i(t_j)$ can be written as

$$y_i(t_j) = (1 + E_j^i) \, Y_i(t_j). \tag{2.22}$$

Substitution of Eq. 2.22 into Eq. 2.11 allows the derivatives $y_i^{(k)}(t_j)$ to be expressed in the following manner:

$$y_i^{(k)}(t_j) = (1 + E_j^i) \, Y_i^{(k)}(t_j) + \sigma_{i,k}(t_j) \qquad \begin{array}{l} k = 1,\ldots,K \\ i = 1,\ldots,m \end{array} \tag{2.23}$$

where $\sigma_{i,k}(t_j)$ is a second order correction term compared to $E_j^i \, Y_i^{(k)}(t_j)$.

For $t = t_{j+1}(t_{j+1} > t_j)$, the exact values are given as in Eq. 2.15 by

$$y_i(t_{j+1}) = \sum_{k=0}^{K} y_i^{(k)}(t_j)(t_{j+1} - t_j)^k/(k!) + R_i(t_{j+1}) \qquad (2.24)$$

whereas the approximate values are given as in Eq. 2.16 by

$$Y_i(t_{j+1}) = \sum_{k=0}^{K} Y_i^{(k)}(t_j)(t_{j+1} - t_j)^k/(k!). \qquad (2.25)$$

Substitution of Eqs. 2.22 and 2.23 into Eq. 2.24 yields to a first approximation

$$y_i(t_{j+1}) = (1 + E_j^i) \sum_{k=0}^{K} Y_i^{(k)}(t_j)(t_{j+1} - t_j)^k/(k!) + R_i(t_{j+1}). \qquad (2.26)$$

Products of small numbers; $\sigma_{i,k}(t_j)(t_{j+1} - t_j)^k/(k!)$ for $k = 1, \ldots, K$; have been neglected in Eq. 2.26.

The substitution of Eq. 2.25 into Eq. 2.26 then yields

$$y_i(t_{j+1}) = (1 + E_j^i) \, Y_i(t_{j+1}) + R_i(t_{j+1}) \qquad (2.27)$$

which by rearrangement becomes

$$\left[y_i(t_{j+1}) - Y_i(t_{j+1}) \right]/Y_i(t_{j+1}) = E_j^i + \left[R_i(t_{j+1})/Y_i(t_{j+1}) \right]. \qquad (2.28)$$

Now, the left-hand side of Eq. 2.28 is E_{j+1}^i by definition (Eq. 2.12). Thus, Eq. 2.28 becomes

$$E_{j+1}^i = E_j^i + \left[R_i(t_{j+1})/Y_i(t_{j+1}) \right]. \qquad (2.29)$$

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By making use of the inequality

$$|a + b| \leq |a| + |b| \tag{2.30}$$

for real numbers a and b, the following result is obtained from Eq. 2.29,

$$\left|E^i_{j+1}\right| \leq \left|E^i_j\right| + \left|\frac{R_i(t_{j+1})}{Y_i(t_{j+1})}\right|. \tag{2.31}$$

Finally, the substitution of Eqs. 2.21 and 2.18 into Eq. 2.31 yields the desired result

$$\left|E^i_{j+1}\right| \leq (j + 1) \, \varepsilon. \tag{2.32}$$

This completes the proof of Eq. 2.13.

To recapitulate, it has been shown that to a first approximation the accumulated relative error after n time steps using continuous analytic continuation is given by Eq. 2.13 for any positive integer n. Thus, the accumulated relative error increases at most linearly with the number of time steps and an upper bound for $\left|E^i_n\right|$ is given by

$$\left|E^i_n\right|_{\text{upper bound}} = n \, \varepsilon. \tag{2.33}$$

The validity of Eq. 2.13 was verified for a few cases (Section 3) by comparing approximate solutions using continuous analytic continuation with some analytic solutions.

From Eq. 2.29, it can be seen that cancellation of errors is possible, if the remainder in the Taylor expansion changes sign during the transient. If K is even, the remainder changes sign when the first derivative of the dependent variable changes sign. If K is odd, the remainder changes sign when the second derivative of the dependent variable changes sign. Thus, for oscillatory transients, the error should accumulate more slowly than for nonoscillatory transients. This was verified for a particular case (Section 3).

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2.4.2. Effect of Order and Error Criterion on Efficiency

Assume the reactor kinetics equations for a particular problem have been solved by continuous analytic continuation up to $t = t_n$ and that the computer time required, exclusive of time required for reading input and writing output, was t_c. The efficiency as used here is defined by

$$\text{efficiency} = t_n/t_c. \tag{2.34}$$

The effects of the order K of the Taylor expansion and of the error criterion ε on the efficiency will now be investigated.

Consider first the effect of order on efficiency. The computing time t_c is equal to the number of time steps n multiplied by the computing time per time step. The computing time per time step is proportional to the number of operations required on the computer per time step, which is in turn, to a first approximation, proportional to K^2. On the other hand, t_n is proportional to n multiplied by the average time step $\overline{\Delta t}$. Thus, the following proportionalities hold

$$t_c \propto nK^2 \tag{2.35}$$

$$t_n \propto n\,\overline{\Delta t}, \tag{2.36}$$

so that

$$t_n/t_c \propto \overline{\Delta t}/K^2. \tag{2.37}$$

The average time step $\overline{\Delta t}$ is given approximately (Section 2.3) by

$$\overline{\Delta t} = [(K + 1)!\ \varepsilon]^{\frac{1}{K+1}} g(K) \tag{2.38}$$

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where g(K) is the average value of

$$\left[\left| N(t) \right| \div \left| N^{(K+1)}(t) \right| \right]^{\frac{1}{K+1}}$$

in the interval $0 \leq t \leq t_n$. Obviously g(K) depends on the particular problem being solved and for each particular problem also depends on t_n. Substitution of Eq. 2.38 into Eq. 2.37 gives

$$t_n/t_c \; \alpha \; f(K) \; g(K), \tag{2.39}$$

where

$$f(K) = [(K + 1)! \; \epsilon]^{\frac{1}{K+1}}/K^2. \tag{2.40}$$

For the simple problem

$$N^{(1)}(t) = \Lambda^{-1} \rho N(t)$$

with ρ equal to a constant, it is seen that

$$N^{(K+1)}(t) = (\Lambda^{-1} \rho)^{K+1} N(t)$$

independent of t so that

$$g(K) = \Lambda/\rho = \text{constant.}$$

Thus, for this simple case

$$t_n/t_c \; \alpha \; f(K).$$

For more complicated problems with reactivity feedback and delayed neutrons, the form of g(K) cannot be determined analytically. However, experience with the method has shown that to a first approximation,

$$g(K) \; \alpha \; 1/K^x \tag{2.41}$$

where x is in the range 0 to 1 for most problems. (x = 0 for the simple problem considered above.) Substitution of Eq. 2.41 into 2.39 yields

$$t_n/t_c \; \alpha \; f(K)/(K)^x \tag{2.42}$$

Figure 2.1 shows the behavior of $f(K)$ as a function of K for several values of the parameter ϵ. Note that $f(K)$ exhibits a maximum which occurs at higher values of K with decreasing ϵ. Figure 2.2 shows the behavior of $f(K)/(K)^x$ as a function of K for $x = 0.5$ and for several values of the parameter ϵ. Note that $f(K)/(K)^{0.5}$ exhibits the same general behavior as $f(K)$ but that the maxima are shifted to smaller values of K. Thus, for problems involving reactivity feedback and delayed neutrons, $g(K)$ shifts the maximum for t_n/t_c to a smaller value of K than that expected from $f(K)$ alone. The qualitative behavior, as exemplified by Figs. 2.1 and 2.2, of the efficiency as a function of K has been verified for a few cases (Section 3).

Consider now the effect of ϵ on the efficiency for a fixed order K. If K is fixed, Eqs. 2.35, 2.36, and 2.38 become

$$t_c \ \alpha \ n \tag{2.43}$$

$$t_n \ \alpha \ n \ \overline{\Delta t} \tag{2.44}$$

$$\overline{\Delta t} \ \alpha \ \epsilon^{\frac{1}{K+1}} . \tag{2.45}$$

Thus

$$t_n/t_c \ \alpha \ \epsilon^{\frac{1}{K+1}} . \tag{2.46}$$

The function $h(\epsilon)/h(10^{-6})$, where

$$h(\epsilon) = \epsilon^{\frac{1}{K+1}} , \tag{2.47}$$

is plotted as a function of ϵ in Fig. 2.3 for several values of the parameter K. It can be seen from the figure that, as the value of K becomes larger, $h(\epsilon)$ becomes less sensitive to changes in ϵ.

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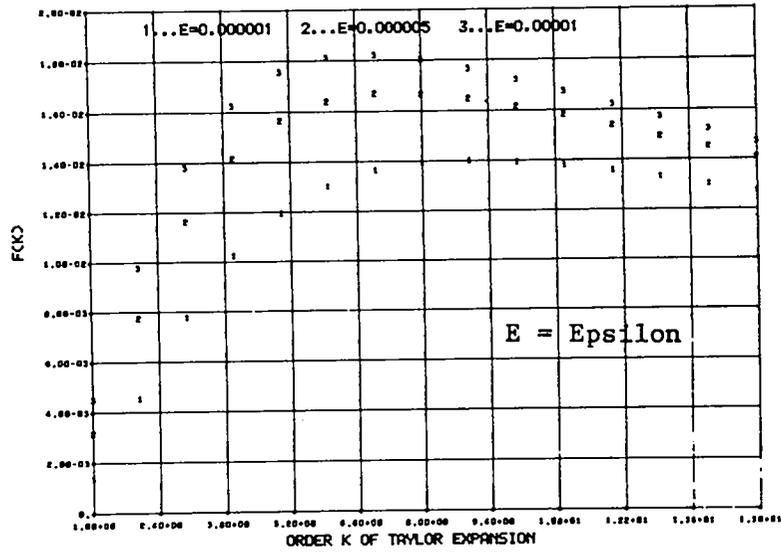
FIG 2.1 PLOT OF F(K) (EQN 2.40) VERSUS K WITH EPSILON AS PARAMETER.
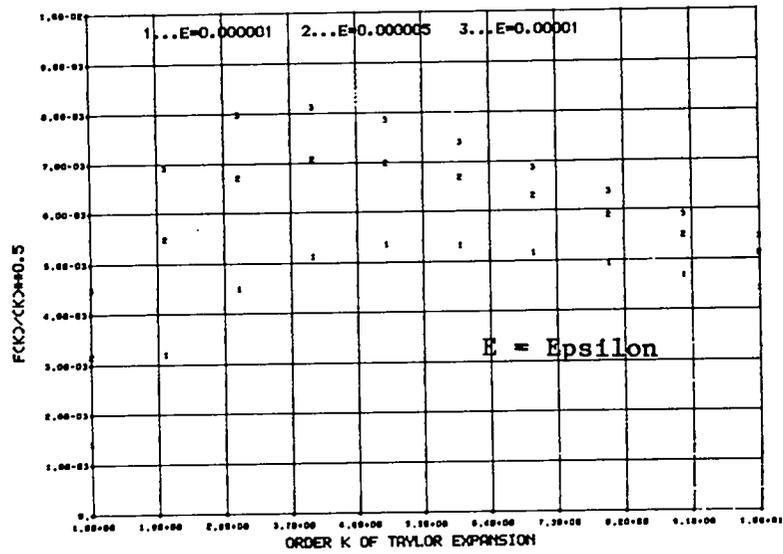
FIG 2.2 PLOT OF F(K)/(K)**0.5 VERSUS K WITH EPSILON AS PARAMETER.

22
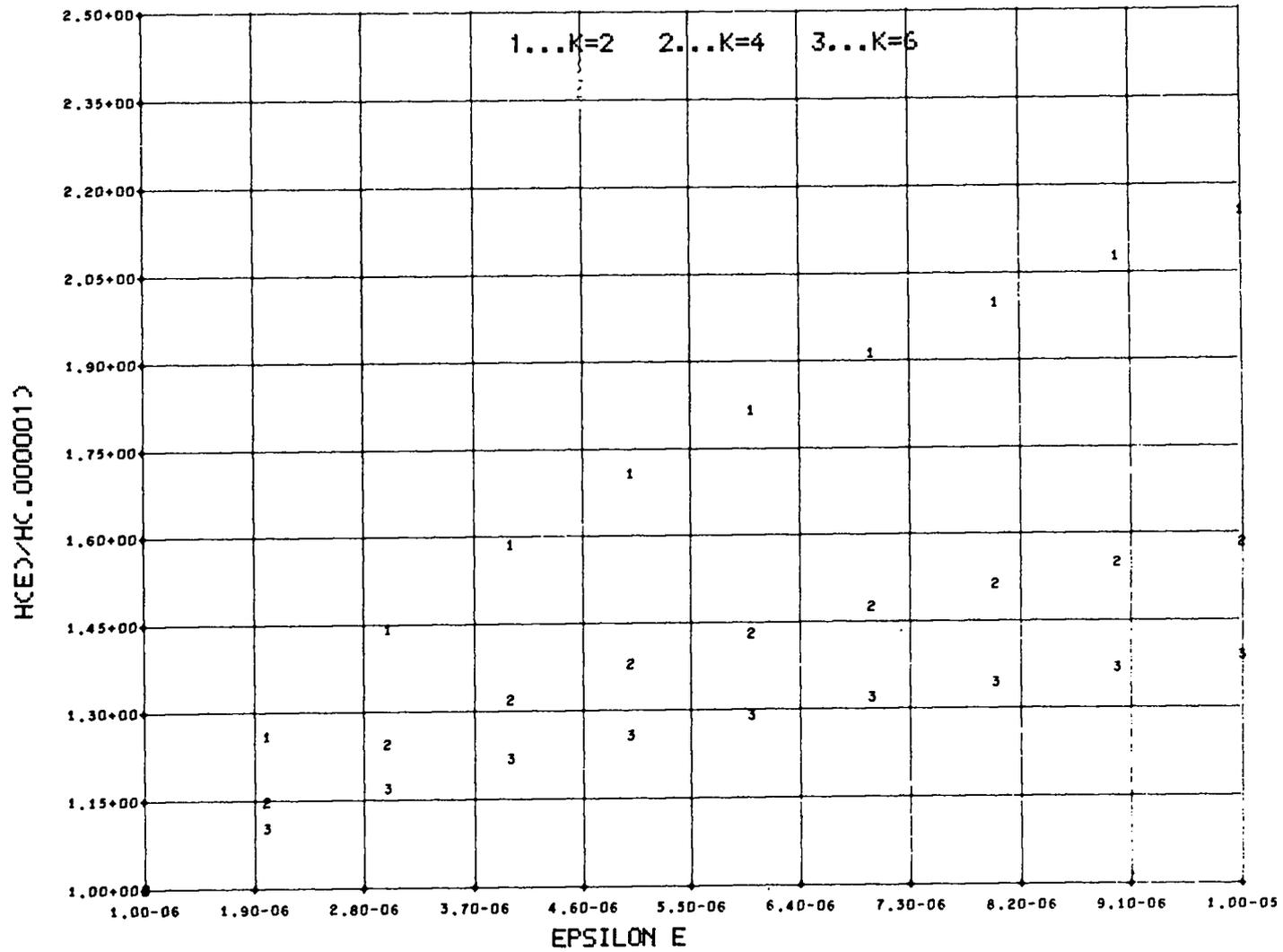
FIG 2.3 PLOT OF H(E)/H(.000001) (SEE EQN 2.47) VS E WITH K AS PARAMETER.

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From the work in Section 2.4, the absolute value of the accumulated relative error $|E_n|$ is proportional to $n\epsilon$. From Eqs. 2.44 and 2.45, n is proportional to

$$\epsilon^{-\frac{1}{K+1}}$$

for a fixed t_n. Hence, $|E_n|$ is proportional to

$$\epsilon^{\frac{K}{K+1}}$$

for a fixed t_n. Thus, the efficiency (Eq. 2.46) is a less sensitive function of ϵ than $|E_n|$. For example, if K = 2, decreasing ϵ by three orders of magnitude should decrease $|E_n|$ (at a fixed t_n) by two orders of magnitude, while t_n/t_c should decrease by only one order of magnitude. Hence, accuracy can be increased by decreasing ϵ without causing a proportionate increase in computing time, the situation becoming more favorable with larger values of K. This property was verified for a few cases in Section 3.

2.5 Variations of Basic Method

For some types of problems, a straightforward application of the basic method may not be the most efficient way, from the viewpoint of computing time, to utilize continuous analytic continuation. In these cases, a modification of the basic method may be faster. Some particular examples are considered below.

2.5.1 Constant Reactivity ($\rho < \beta$) with Delayed Neutrons and Sources

The first example considered is the case of constant reactivity (in particular, $\rho < \beta$) with delayed neutrons and an extraneous source. This case is described by the equations,

$$N^{(1)}(t) = \Lambda^{-1}(\rho - \beta) N(t) + \sum_{i=1}^{I} \lambda_i C_i(t) + S \qquad (2.48)$$

$$C_i^{(1)}(t) = \Lambda^{-1} \beta_i N(t) - \lambda_i C_i(t) \qquad i = 1, \ldots, I \qquad (2.49)$$

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where ρ is a constant which is smaller than β. For this case, the response of the reactor is controlled by the delayed neutrons, except for the initial prompt jump.

In the basic method, the time step is determined by $N(t)$. However, it is possible to modify the basic method in the case under consideration, so that the time step is determined by the delayed neutron precursors $C_i(t)$. This results in much larger time steps than would be allowed by the basic method for the same accuracy. A description of the variation in the basic method follows:

The values of $C_i(t)$ at $t = t_{j+1}$ are determined as usual from,

$$C_i(t_{j+1}) = \sum_{k=0}^{K} C_i^{(k)}(t_j)(t_{j+1} - t_j)^k/(k!) \qquad i=1, \ldots, I \qquad (2.50)$$

but $t_{j+1} - t_j$ is calculated from the time step criterion (Section 2.3) using the delayed neutron precursor which is varying the most rapidly. Since the time step determined in this manner can be much larger than that which would be required for a Taylor expansion of $N(t)$ to the same accuracy, $N(t_{j+1})$ is calculated explicitly from $C_i(t_j)$ and $N(t_j)$, instead of from an equation similar to Eq. 2.50. This procedure requires that the higher derivatives of $C_i(t)$ be decoupled from those of $N(t)$.

The first order derivative in Eq. 2.50 is calculated from Eq. 2.49. Higher order derivatives in Eq. 2.50 are obtained, as follows: Differentiate Eq. 2.49 and use Eqs. 2.48 and 2.49 in the result to eliminate $N(t)$. The result is

$$C_i^{(2)}(t) = a_i C_i^{(1)}(t) + b_i C_i(t) + d_i \sum_{j=1}^{I} \lambda_j C_j(t) + d_i S \qquad (2.51)$$

$$i = 1. \ldots. I$$

where

$$a_i = \Lambda^{-1}(\rho - \beta) - \lambda_i$$
$$b_i = \Lambda^{-1}(\rho - \beta) \lambda_i$$
$$d_i = \Lambda^{-1} \beta_i.$$

Since Eq. 2.51 is linear, it is seen by inspection that

$$C_i^{(k+2)}(t) = a_i \, C_i^{(k+1)}(t) + b_i \, C_i^{(k)}(t) + d_i \sum_{j=1}^{I} \lambda_j \, C_j^{(k)}(t) \qquad (2.52)$$

$$i = 1, \ldots, I$$

for $k \geq 0$.

Equation 2.48 can be integrated by the use of the integrating factor $\exp[-\Lambda^{-1}(\rho - \beta)t]$. The result is

$$N(t_{j+1}) = N(t_j) \, e^{\omega \Delta t} + e^{\omega \Delta t} \sum_{i=1}^{I} \lambda_i \int_{t_j}^{t_{j+1}} C_i(t) \, e^{-\omega(t - t_j)} \, dt$$

$$+ \frac{S}{\omega} (e^{\omega \Delta t} - 1) \qquad (2.53)$$

where

$$\omega = \Lambda^{-1}(\rho - \beta)$$

$$\Delta t = t_{j+1} - t_j$$

The delayed neutron precursor levels in the interval $t_j \leq t \leq t_{j+1}$ are given by (Eq. 2.50)

$$C_i(t) = \sum_{k=0}^{K} C_i^{(k)}(t_j)(t - t_j)^k / (k!) \qquad t_j \leq t \leq t_{j+1} \qquad (2.54)$$

$$i = 1, \ldots, I.$$

Thus the integral in the second term on the right-hand side of Eq. 2.53 becomes,

$$\int_{t_j}^{t_{j+1}} C_i(t) \, e^{-\omega(t - t_j)} \, dt = \sum_{k=0}^{K} \frac{C_i^{(k)}(t_j)}{k!} \int_{t_j}^{t_{j+1}} e^{-\omega(t - t_j)} (t - t_j)^k \, dt$$

$$= \sum_{k=0}^{K} C_i^{(k)}(t_j) \, In(k, \Delta t) \qquad (2.55)$$

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where

$$In(k,\Delta t) = \int_0^{\Delta t} e^{-\omega x} x^k \, dx \qquad (2.56)$$

and

$$x = t - t_j.$$

The integral in Eq. 2.56 is a standard one and is equal to the finite series:

$$\frac{-e^{-\omega x}}{\omega^{k+1}} \left[(\omega x)^k + k(\omega x)^{k-1} + k(k-1)(\omega x)^{k-2} + \ldots + k! \right].$$

By evaluating this finite series at the limits 0 and Δt, obtain

$$In(k,\Delta t) = \frac{-e^{-\omega \Delta t}}{\omega^{k+1}} \left[(\omega \Delta t)^k + k(\omega \Delta t)^{k-1} + k(k-1)(\omega \Delta t)^{k-2} \right.$$
$$\left. + \ldots + k! \right] + \frac{k!}{\omega^{k+1}} \qquad (2.57)$$

Substitution of Eq. 2.57 into Eq. 2.55 then yields

$$\int_{t_j}^{t_{j+1}} C_i(t) \, e^{-\omega(t-t_j)} \, dt = \sum_{k=0}^{K} \frac{C_i^{(k)}(t_j)}{\omega^{k+1}} \cdot$$
$$\left\{ 1 - e^{-\omega \Delta t} \left[\frac{(\omega \Delta t)^k}{k!} + \frac{(\omega \Delta t)^{k-1}}{(k-1)!} + \frac{(\omega \Delta t)^{k-2}}{(k-2)!} + \ldots + 1 \right] \right\}. \qquad (2.58)$$

From Eqs. 2.58 and 2.53 is obtained

$$N(t_{j+1}) = N(t_j) \, e^{\omega \Delta t} + \sum_{i=1}^{I} \lambda_i \sum_{k=0}^{K} \frac{C_i^{(k)}(t_j)}{\omega^{k+1}} \cdot$$

$$\left\{ e^{\omega \Delta t} - \left[\frac{(\omega \Delta t)^k}{k!} + \frac{(\omega \Delta t)^{k-1}}{(k-1)!} + \frac{(\omega \Delta t)^{k-2}}{(k-2)!} + \ldots + 1 \right] \right\} + \frac{S}{\omega}(e^{\omega \Delta t} - 1).$$

$$(2.59)$$

The integral of the neutron level between t_j and t_{j+1} is obtained from Eq. 2.59 which holds in the interval $t_j \leq t \leq t_{j+1}$. The result is

$$\int_{t_j}^{t_{j+1}} N(t') \, dt' = \frac{N(t_j)}{\omega} (e^{\omega \Delta t} - 1) + \sum_{i=1}^{I} \lambda_i \sum_{k=0}^{K} \frac{c_i^{(k)}(t_j)}{\omega^{k+2}}$$

$$\left\{ e^{\omega \Delta t} - 1 - \left[\frac{(\omega \Delta t)^{k+1}}{(k+1)!} + \frac{(\omega \Delta t)^k}{k!} + \frac{(\omega \Delta t)^{k-1}}{(k-1)!} + \ldots + \omega \Delta t \right] \right\}$$

$$+ \frac{S}{\omega^2} (e^{\omega \Delta t} - \omega \Delta t - 1). \tag{2.60}$$

In order to compute the instantaneous inverse period at $t = t_{j+1}$, $N^{(1)}(t_{j+1})$ is required. Using Eq. 2.59, it is found that

$$N^{(1)}(t_{j+1}) = \omega N(t_j) e^{\omega \Delta t} + \sum_{i=1}^{I} \lambda_i \sum_{k=0}^{K} \frac{c_i^{(k)}(t_j)}{\omega^k}$$

$$\left\{ e^{\omega \Delta t} - \left[\frac{(\omega \Delta t)^{k-1}}{(k-1)!} + \frac{(\omega \Delta t)^{k-2}}{(k-2)!} + \ldots + 1 \right] \right\} + S e^{\omega \Delta t}. \tag{2.61}$$

For problems in which this variation of the basic method is applicable, the time steps can be as much as two or more orders of magnitude larger than those obtained with a straightforward application of the basic method. Thus, the computer program (Appendix D) has been written to solve problems of the type under consideration using this variation of the basic method.

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2.5.2 Near Exponential Behavior

In most reactor transients, $N(t)$ and $C_i(t)$ rise or fall in an exponential or near exponential manner over a part or parts of the transient. In these regions, the variation of the basic method discussed below may be more efficient from the viewpoint of computing time.

The functions $N(t)$ and $C_i(t)$ are rewritten as

$$N(t) = e^{\omega_n(t-t_j)} R(t)$$

$$C_i(t) = e^{\omega_n(t-t_j)} W_i(t) \qquad t_j \leq t \leq t_{j+1} \qquad (2.62)$$

where

$$\omega_n = N^{(1)}(t_j)/N(t_j). \qquad (2.63)$$

Thus, the functions $R(t)$ and $W_i(t)$ give the residual variation remaining in $N(t)$ and $C_i(t)$ after the exponential behavior has been extracted. To determine the governing equations for $R(t)$ and $W_i(t)$, Eqs. 2.62 are substituted into the reactor kinetics equations

$$N^{(1)}(t) = \Lambda^{-1}[\rho(t) - \beta] N(t) + \sum_{i=1}^{I} \lambda_i C_i(t) + S$$

$$C_i^{(1)}(t) = \Lambda^{-1} \beta_i N(t) - \lambda_i C_i(t) \qquad i = 1,\ldots,I \qquad (2.64)$$

yielding

$$R^{(1)}(t) = \Lambda^{-1}[\rho(t) - \beta - \Lambda\omega_n] R(t) + \sum_{i=1}^{I} \lambda_i W_i(t) + Se^{-\omega_n(t-t_j)}$$

$$t_j \leq t \leq t_{j+1}$$

$$W_i^{(1)}(t) = \Lambda^{-1} \beta_i R(t) - (\lambda_i + \omega_n) W_i(t)$$

$$i = 1,\ldots,I. \qquad (2.65)$$

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The higher derivatives of $R(t)$ and $W_i(t)$ are obtained by successive differentiation of Eqs. 2.65. From Eqs. 2.62, it can be seen that the values of $R(t)$ and $W_i(t)$ at the beginning of the interval, that is at $t = t_j$, are given by

$$R(t_j) = N(t_j)$$

$$W_i(t_j) = C_i(t_j). \qquad (2.66)$$

The equations defining $\rho(t)$ remain unchanged, except that $N(t)$ in the heat balance equations is replaced by $\exp[\omega_n(t - t_j)] R(t)$.

For exponential or near exponential variation of $N(t)$ and $C_i(t)$ in the interval $t_j \leq t \leq t_{j+1}$, $R(t)$ and $W_i(t)$ are slowly varying functions of time. By applying continuous analytic continuation to Eqs. 2.65 instead of to Eqs. 2.64, larger time steps will be allowed for the same relative truncation error in these portions of the transient. However, in regions where the response is far from exponential, the use of Eqs. 2.65 may give smaller time steps than Eqs. 2.64. Also, the expressions for the integral of $N(t)$ between t_j and t_{j+1} and for the higher derivatives of the heat balance equations contain more terms using Eqs. 2.65 than with Eqs. 2.64. Thus, the use of Eqs. 2.65 is expected to yield shorter computing times only when a major portion of the excursion is exponential or near exponential in character.

The procedure for solving Eqs. 2.65 and the associated equations which describe $\rho(t)$ in the interval $t_j \leq t \leq t_{j+1}$ is basically the same as described previously for Eqs. 2.64. After the values of $R(t_{j+1})$ and $W_i(t_{j+1})$ have been obtained from the Taylor expansion about $t = t_j$, the values of $N(t_{j+1})$ and $C_i(t_{j+1})$ are obtained from Eqs. 2.62 by setting $t = t_{j+1}$.

2.5.3 Miscellaneous Variations

Two miscellaneous variations of the basic method which can increase the efficiency of continuous analytic continuation are mentioned briefly. .

First, consider excursions starting from essentially zero power level. For these problems, the feedback equations can be bypassed until sufficient energy has been generated to increase the temperature of the reactor. This does not occur until the power level has increased by several decades above the initial power level.

Secondly, since the time step required by $N(t)$ and $C_i(t)$ is much smaller than that required by the feedback equations, the efficiency of the method can be increased by doing several outer iterations on the neutronic equations per outer iteration on the feedback equations. An alternative to this approach is to use a lower order K on the feedback equations than that used on the neutronic equations.

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3. RESULTS

The method of continuous analytic continuation was applied to a variety of reactor kinetics problems. As a check on the accuracy of the method, comparisons were made with analytic solutions in a few special (mostly linear) cases for which analytic solutions exist. Comparisons were also made with results from another numerical method for a few nonlinear cases, as well as with some experimental transients reported in the literature.

Since the reactor kinetics equations can be expressed in two different forms (Appendix A) -- one based on the prompt neutron generation time and the other on the prompt neutron lifetime -- comparisons of solutions using each of these two forms were made in order to study the effect of representation of the reactor kinetics equations.

The efficiency (Section 2.4.2) as a function of K and ϵ was obtained for those problems which had a computing time of at least a few tenths of a minute, since time is kept by the computer only within an accuracy of ± 0.01 minute. The time can be obtained during execution of the program by use of the subroutine CLOCK (Appendix D). The computing time t_c used in calculating the efficiency is obtained by calling the subroutine CLOCK at the start of the problem (after all input data have been read in) and at the end of the problem (before writing and/or punching output).

3.1 Comparisons with Analytic Solutions

Six cases for which analytic solutions exist were computed using continuous analytic continuation. For all of these cases except one, the differential equations are linear. The approximate results were compared with the analytic solutions for each variable occurring in the equations. The variables

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$$\int_0^t N(t') \, dt'$$

and

$$\alpha(t) = N^{(1)}(t)/N(t)$$

were included in the comparison for most of the cases. The integrated neutron level is computed using continuous analytic continuation in the following manner: From Eq. 2.4 $N(t)$ is given in the interval $t_{n-1} \leq t \leq t_n$ by

$$N(t) = \sum_{k=0}^{K} N^{(k)}(t_{n-1})(t - t_{n-1})^k \Big/ (k!) \qquad t_{n-1} \leq t \leq t_n \qquad (3.1)$$

so that

$$\int_{t_{n-1}}^{t_n} N(t') \, dt' = \sum_{k=0}^{K} N^{(k)}(t_{n-1})(t_n - t_{n-1})^{k+1} \Big/ (k+1)! \qquad (3.2)$$

Thus, the integral

$$\int_0^t N(t') \, dt'$$

is obtained by summing Eq. 3.2 over all time steps in the interval 0 to t.

It should be mentioned that in all problems considered in this and subsequent sections, only a fraction of the points from the approximate solutions were plotted. Analytic results were computed at corresponding points to allow calculation of the accumulated error. In Section 2.4, it was shown that to first order, the absolute value of the accumulated relative (or fractional) error for variable y_i is less than or equal to $n\varepsilon$ (Eq. 2.13), so that the upper limit is $n\varepsilon$ (Eq. 2.33). In this section, the actual accumulated relative error for the various cases is compared with Eq. 2.33 in order to verify that Eq. 2.13 holds.

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3.1.1 Case 1: Step Change in Reactivity; No Delayed Neutrons, Source, or Feedback

The simplest problem for which analytic solutions exist is that of a step change in reactivity with no delayed neutrons, source, or feedback. This case is represented by the equation

$$N^{(1)}(t) = \Lambda^{-1} \rho_o N(t) \tag{3.3}$$

which by inspection has the solutions

$$N(t)/N(0) = e^{\Lambda^{-1}\rho_o t} \tag{3.4}$$

$$\int_0^t N(t') \, dt' = \Lambda \rho_o^{-1} N(0) \left[e^{\Lambda^{-1}\rho_o t} - 1 \right]. \tag{3.5}$$

Approximate results using continuous analytic continuation with $K = 6$ and $\epsilon = 10^{-6}$ are compared with analytic results in Figs. 3.1A and 3.1B for the case $\Lambda = 10^{-5}$ sec, $\rho_o = 6.4 \times 10^{-4}$, and $N(0) = 1.0$. As can be seen from these figures, the agreement between the approximate and analytic results is excellent. The actual accumulated relative error in $N(t)$ for this case is compared with Eq. 2.33 in Fig. 3.1C, and the results are in agreement with Eq. 2.13. Figure 3.1D shows the actual relative error in $N(t)$ and the upper limit from Eq. 2.33 at a fixed point ($t = 0.5$ sec) in the transient as a function of ϵ with K as parameter. Again, it is seen that the relative (or fractional) error satisfies Eq. 2.13. In Section 2.4.2, it was shown that at a fixed point in the transient the accumulated relative error is proportional to $\epsilon^{K/K+1}$. This is verified in Fig. 3.1D for the case under consideration.

Because the computing time was only of the order of 0.01 minute for this problem even on carrying out the solution to $N(t)/N(0) = 10^{38}$, it was not possible to study the effects of K and ϵ on the efficiency.
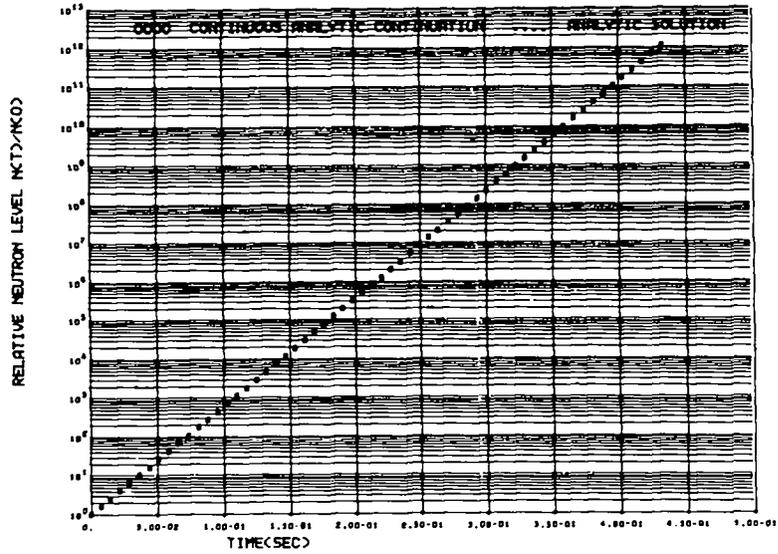
FIG 3.1A COMPARISON OF APPROX AND ANALYTIC RESULTS FOR CASE 1.
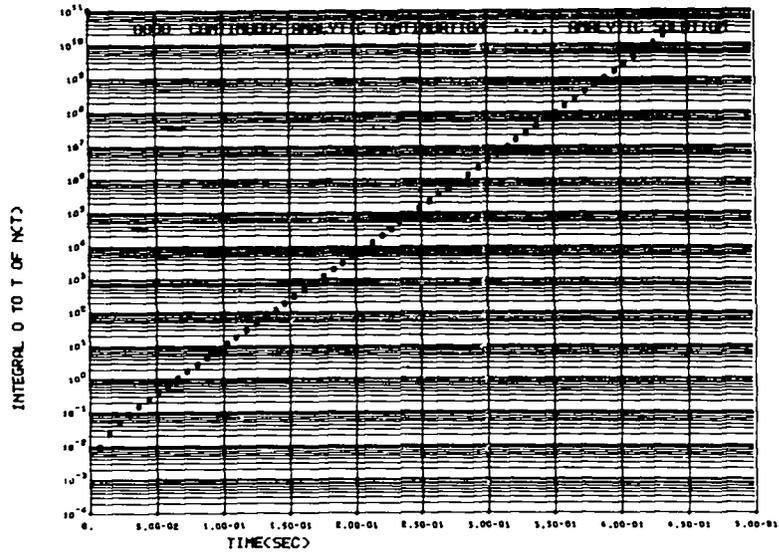
FIG 3.1B COMPARISON OF APPROX AND ANALYTIC RESULTS FOR CASE 1.

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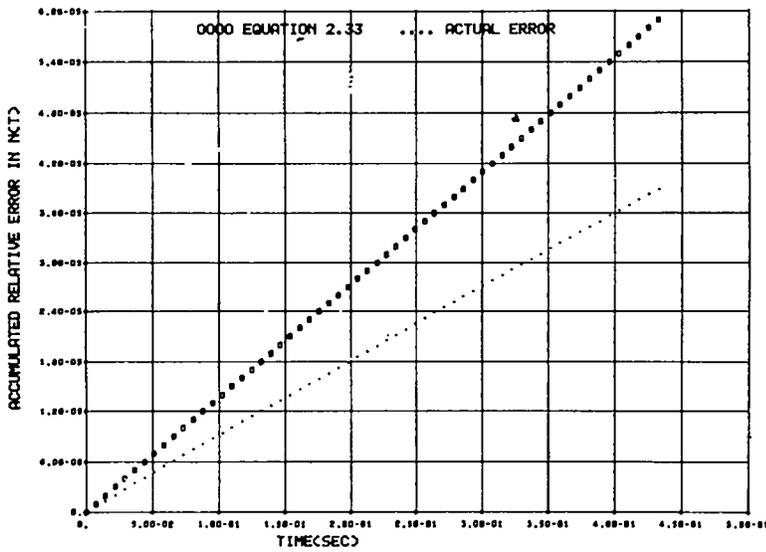
FIG 3.1C COMPARISON OF ACTUAL RELATIVE ERROR WITH EQN 2.33 FOR CASE 1.
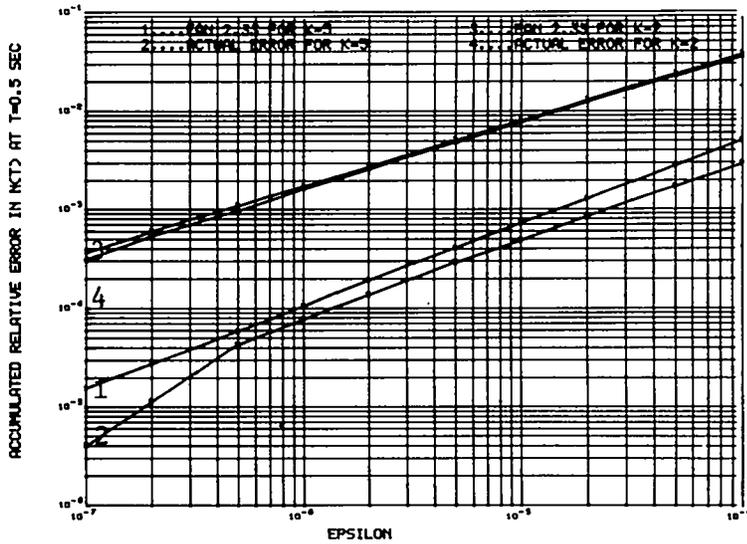
FIG 3.1D ACCUMULATED RELATIVE ERROR FOR N(T) AS A FUNCTION OF EPSILON
WITH K AS PARAMETER. COMPUTED AT T=0.5 SEC FOR CASE 1.

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3.1.2. Case 2: Ramp Input in Reactivity; No Delayed Neutrons, Source, or Feedback

Another simple problem for which analytic solutions exist is that of a ramp input in reactivity with no delayed neutrons, source, or feedback. This problem is represented by the equations

$$N^{(1)}(t) = \Lambda^{-1} \rho(t) N(t) \tag{3.6}$$

$$\rho(t) = bt \tag{3.7}$$

where b is a constant. The solutions to this problem are

$$N(t)/N(0) = e^{bt^2/2\Lambda} \tag{3.8}$$

$$\int_0^t N(t') \, dt' = N(0)t \sum_{n=0}^{\infty} \frac{(at^2)^n}{n!(2n+1)} \tag{3.9}$$

$$\alpha(t) = N^{(1)}(t)/N(t) = b \Lambda^{-1} t \tag{3.10}$$

where $\alpha(t)$ is the instantaneous inverse period and $a = b/2\Lambda$. Equation 3.9 is obtained by expanding the exponential in an infinite series and integrating term by term.

Comparisons of approximate results using continuous analytic continuation with $K = 4$ and $\epsilon = 10^{-6}$ are made with the analytic results in Figs. 3.2A through 3.2D for the case $\Lambda = 8 \times 10^{-9}$ sec, $b = 2.1 \times 10^{-3}$, and $N(0) = 1.0$. It can be seen from the figures that the approximate results are in excellent agreement with the analytic results. In Fig. 3.2E, the actual accumulated relative error is compared with Eq. 2.33; the results are in agreement with Eq. 2.13. The actual relative error in $N(t)$ and the upper limit predicted by Eq. 2.33 at a fixed point $(t = 1.3 \times 10^{-2}$ sec) in the transient are shown in Fig. 3.2F as functions of ϵ with K as parameter. As expected, the figure shows that the accumulated relative error satisfies Eq. 2.13 and that it is proportional to $\epsilon^{K/K+1}$ (Section 2.4.2).
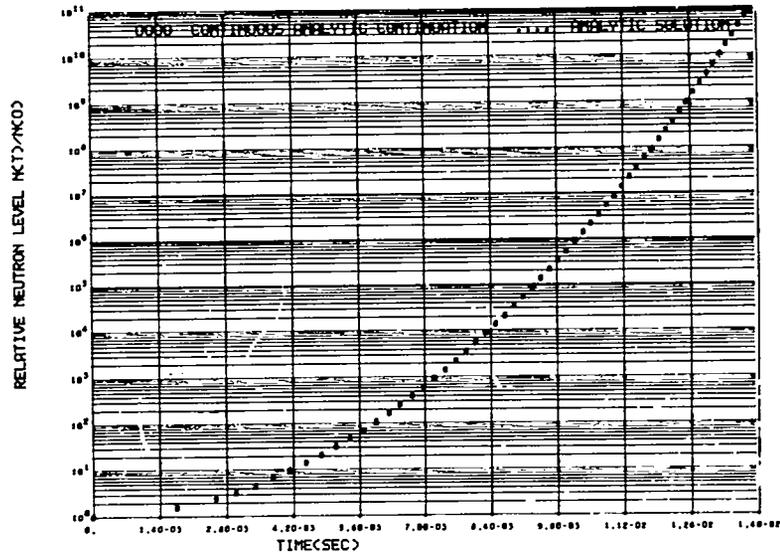
FIG 3.2A COMPARISON OF APPROX AND ANALYTIC RESULTS FOR CASE 2.
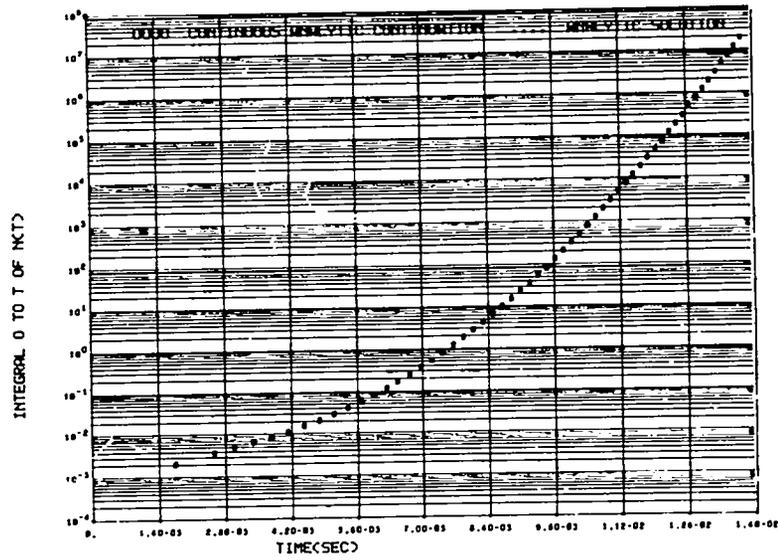
FIG 3.2B COMPARISON OF APPROX AND ANALYTIC RESULTS FOR CASE 2.

38
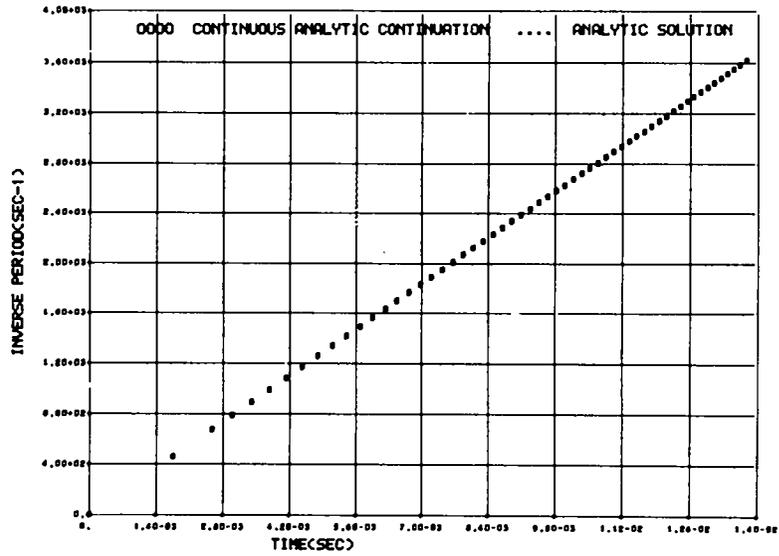
FIG 3.2C COMPARISON OF APPROX AND ANALYTIC RESULTS FOR CASE 2.
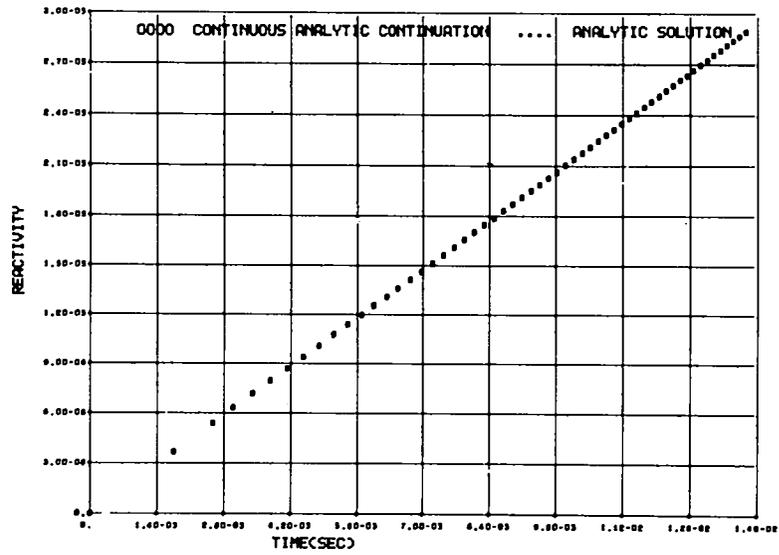
FIG 3.2D COMPARISON OF APPROX AND ANALYTIC RESULTS FOR CASE 2.
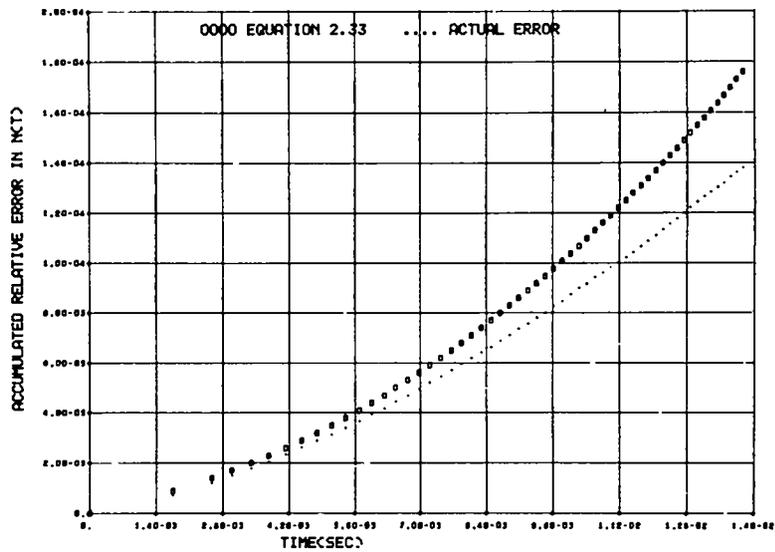
FIG 3.2E   COMPARISON OF ACTUAL RELATIVE ERROR WITH EQN 2.33 FOR CASE 2.
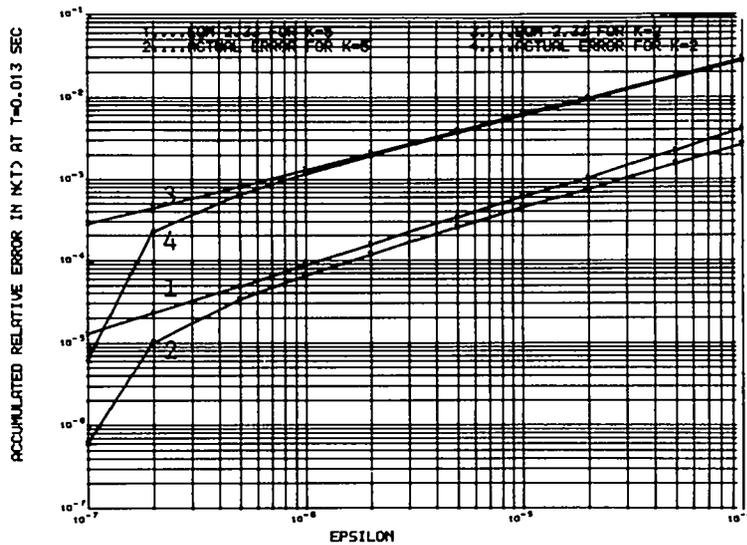
FIG 3.2F   ACCUMULATED RELATIVE ERROR FOR N(T) AS A FUNCTION OF EPSILON
WITH K AS PARAMETER, COMPUTED AT T=0.013 SEC FOR CASE 2.

For this problem, the odd derivatives of N(t) at t = 0 are zero. Thus, the initial value of the first time step cannot be computed from Eq. 2.10 when K is even. The results in Figs. 3.2A through 3.2E indicate that the procedure (Section 2.3.2) used to eliminate this difficulty is satisfactory.

As in Case 1, the computing time for this problem was only about 0.01 minute, and thus, it was not possible to investigate the efficiency as a function of K and ε.

3.1.3. Case 3: Step Change in Reactivity with Thermal Feedback; No Delayed Neutrons or Source

One of the few nonlinear problems for which analytic solutions exist is that of a step change in reactivity with thermal feedback and no delayed neutrons or source. The following equations apply in this case:

$$N^{(1)}(t) = \Lambda^{-1} \rho(t) N(t) \tag{3.11}$$

$$\rho(t) = a + b[T(t) - T(0)] \tag{3.12}$$

$$T^{(1)}(t) = H N(t) \tag{3.13}$$

where

T(t) is the temperature of the reactor,

    a is the initial value of $\rho(t)$,

    b is the temperature coefficient of reactivity, and

    H is a constant involving the heat capacity and a conversion factor from neutron level to power level.

Equation 3.13 applies to the adiabatic case, since heat losses are not taken into account. This equation can be integrated directly to yield

$$T(t) - T(0) = H \int_0^t N(t') \, dt' . \tag{3.14}$$

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By substitution of Eq. 3.14 into 3.12, one obtains

$$\rho(t) = a + b H \int_0^t N(t') \, dt', \qquad (3.15)$$

which when substituted into Eq. 3.11 yields

$$N^{(1)}(t) = \Lambda^{-1} a N(t) + \Lambda^{-1} bH N(t) \int_0^t N(t') \, dt' . \qquad (3.16)$$

Equation 3.16 is a nonlinear integro-differential equation in $N(t)$.

The solution to the system of Eqs. 3.11 through 3.13 is given in Appendix C. The results are

$$\frac{N(t)}{N(0)} = \left(\frac{r_1 - r_2}{r_1 - r_2 \, e^{\omega t}} \right)^2 e^{\omega t} \qquad (3.17)$$

$$\int_0^t N(t') \, dt' = \frac{N(0)(r_1 - r_2)}{\omega} \left(\frac{e^{\omega t} - 1}{r_1 - r_2 \, e^{\omega t}} \right) \qquad (3.18)$$

$$\rho(t) = a + \frac{b \, r_1 \, r_2 (e^{\omega t} - 1)}{r_1 - r_2 \, e^{\omega t}} \qquad (3.19)$$

$$T(t) = T(0) + \frac{r_1 \, r_2 (e^{\omega t} - 1)}{r_1 - r_2 \, e^{\omega t}} \qquad (3.20)$$

$$\alpha(t) = \frac{N^{(1)}(t)}{N(t)} = \Lambda^{-1} a + \frac{\Lambda^{-1} b \, r_1 \, r_2 (e^{\omega t} - 1)}{r_1 - r_2 \, e^{\omega t}} \qquad (3.21)$$

where

$$\omega = \Lambda^{-1} \left[a^2 - 2b \, H \, \Lambda \, N(0) \right]^{\frac{1}{2}}$$

$$r_1 = (a + \Lambda\omega)/b$$

$$r_2 = (a - \Lambda\omega)/b.$$

Approximate results using continuous analytic continuation with $K = 4$ and $\varepsilon = 10^{-6}$ are compared with the analytic solutions in Figs. 3.3A through 3.3E. The comparison is for the case $\Lambda = 10^{-3}$ sec, $a = 10^{-2}$, $b = -10^{-4}/°K$, $H = 10^{-14}$ °K/sec, $N(0) = 10^5$, and $T(0) = 303$ °K. It can be seen from these figures that the approximate results are in excellent agreement with the analytic solutions.

The actual accumulated relative error for $N(t)$ and $T(t)$ is compared with Eq. 2.33 in Fig. 3.3F. Since $\rho(t)$ is zero near $t = 2.8$ sec, the relative error for $\rho(t)$ is not defined in the vicinity of this point. However, the accumulated relative error for $\rho(t)$ in other portions of the transient is almost identical to that for $T(t)$. As expected, the results in Fig. 3.3F are in agreement with Eq. 2.13. It should be recalled (Section 2.3) that a common time step for all the dependent variables is determined from the behavior of $N(t)$. The assumption was made in Section 2.3 that the use of this time step would guarantee that the other dependent variables would also satisfy the truncation error criterion (Eq. 2.5). That this is the case for the problem under consideration can be seen from Fig. 3.3F which shows that the error accumulates more slowly for $T(t)$ than for $N(t)$.

Another interesting point concerning Fig. 3.3F should be mentioned. The time step is determined in a manner (Section 2.3) such that the relative or fractional truncation error is $\leq \varepsilon$ for each time step. Thus, the truncation error at the peak in the neutron level (Fig. 3.3A), where $N(t)/N(0) \doteq 5 \times 10^{11}$, is much larger than at the tail end of the transient where $N(t)/N(0)$ is decreasing rapidly. Figure 3.3F shows, however, that
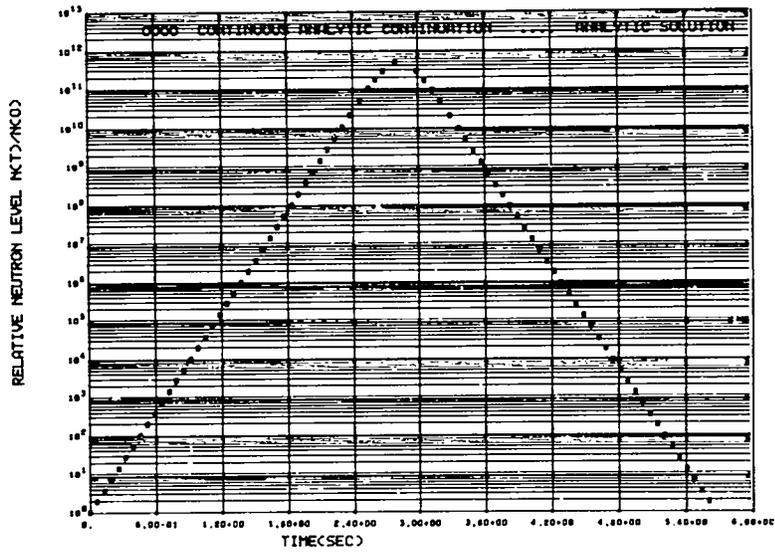
FIG 3.3A COMPARISON OF APPROX AND ANALYTIC RESULTS FOR CASE 3.
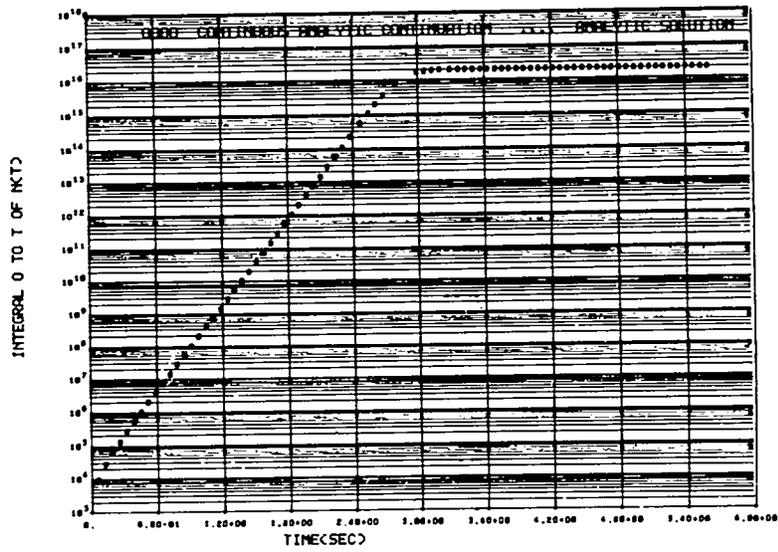
FIG 3.3B COMPARISON OF APPROX AND ANALYTIC RESULTS FOR CASE 3.

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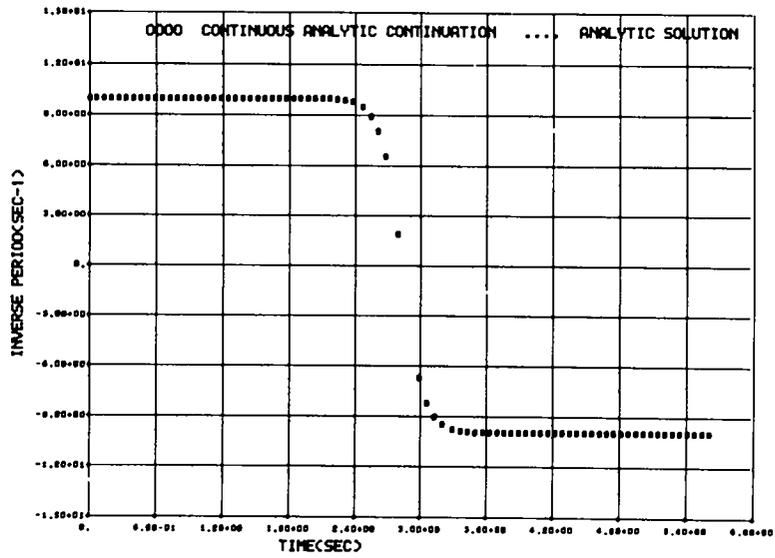
FIG 3.3C COMPARISON OF APPROX AND ANALYTIC RESULTS FOR CASE 3.
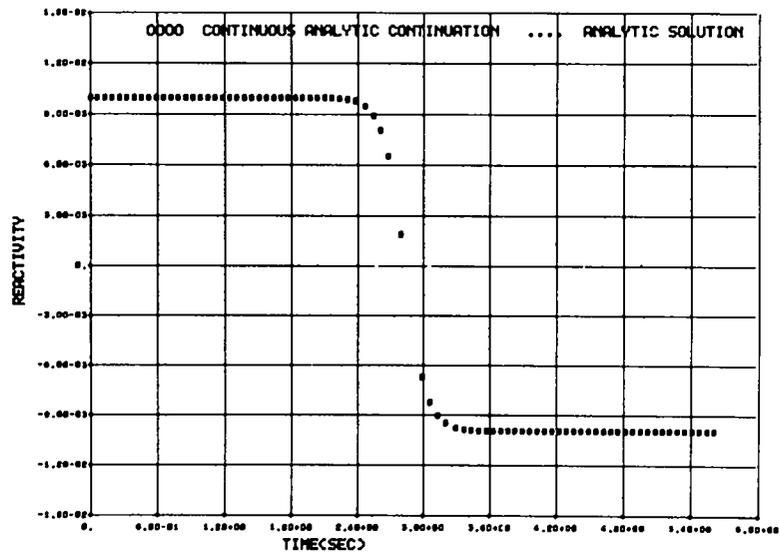
FIG 3.3D COMPARISON OF APPROX AND ANALYTIC RESULTS FOR CASE 3.

45
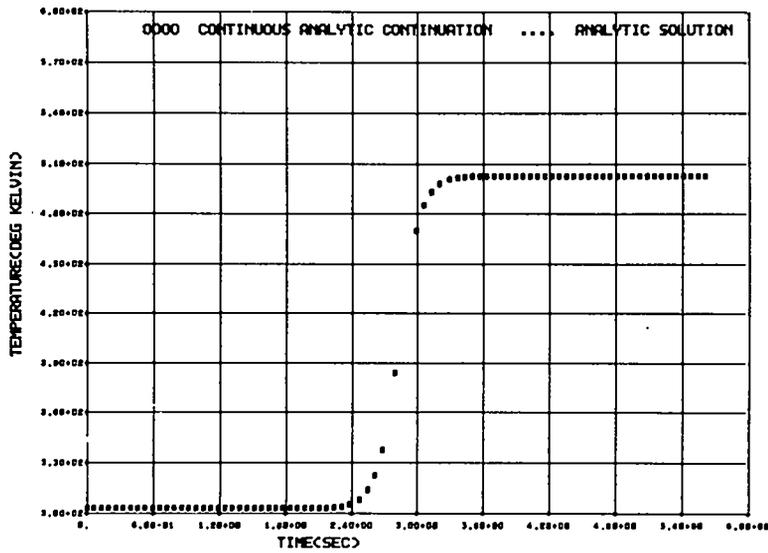
FIG 3.3E COMPARISON OF APPROX AND ANALYTIC RESULTS FOR CASE 3.
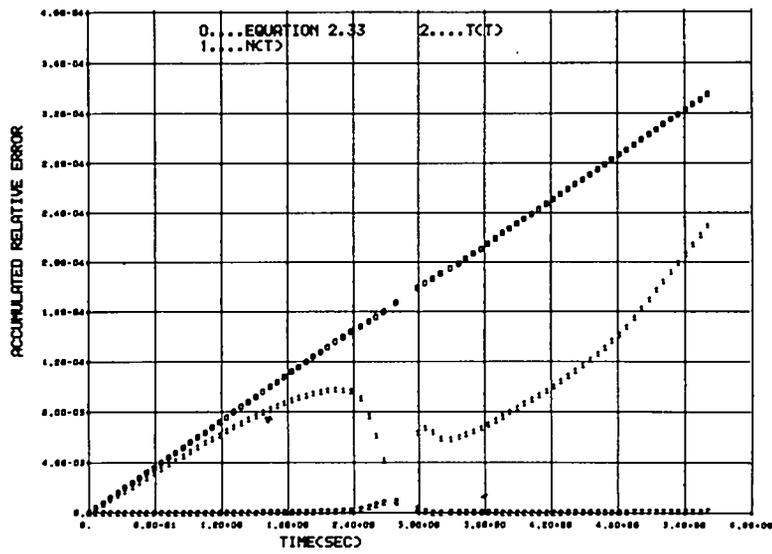

FIG 3.3F COMPARISON OF ACTUAL RELATIVE ERROR FOR N(T)
AND T(T) WITH EQN 2.33 FOR CASE 3.

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Eq. 2.13 for the accumulated relative error continues to hold for the tail end of the transient. The reason for this is that the remainder in the Taylor expansion for N(t) changes sign in the vicinity of the peak, resulting in cancellation of most of the error accumulated up to that point. This is evident in Fig. 3.3F from the dip in the accumulated relative error for N(t).

The actual relative error in N(t) and the upper limit from Eq. 2.33 are plotted as a function of ϵ with K as parameter in Fig. 3.3G at a fixed point (t = 5.6 sec) in the transient. The accumulated error satisfies Eq. 2.13 and is proportional to $\epsilon^{K/K+1}$ as expected (Section 2.4.2).

Again, the computing time for this problem was too short ($\approx$0.03 min) to allow investigation of the efficiency as a function of K and ϵ.

3.1.4. Case 4: Sinusoidal Variation in Reactivity; No Delayed Neutrons, Source, or Feedback

Another problem for which analytic solutions are available is that of a sinusoidal variation in reactivity with no delayed neutrons, source, or feedback. This problem is represented by the equations

$$N^{(1)}(t) = \Lambda^{-1} \rho(t) N(t) \tag{3.22}$$

$$\rho(t) = \rho_o \sin \omega t \tag{3.23}$$

where ρ_o and ω are constants.

By inspection, the solution to this problem is

$$\frac{N(t)}{N(0)} = \exp\left[\frac{\rho_o}{\Lambda\omega}(1 - \cos \omega t)\right]$$

$$= \exp\left(\frac{2\rho_o}{\Lambda\omega} \sin^2 \frac{\omega t}{2}\right) \tag{3.24}$$

$$\alpha(t) = N^{(1)}(t)/N(t) = \Lambda^{-1} \rho_o \sin \omega t. \tag{3.25}$$
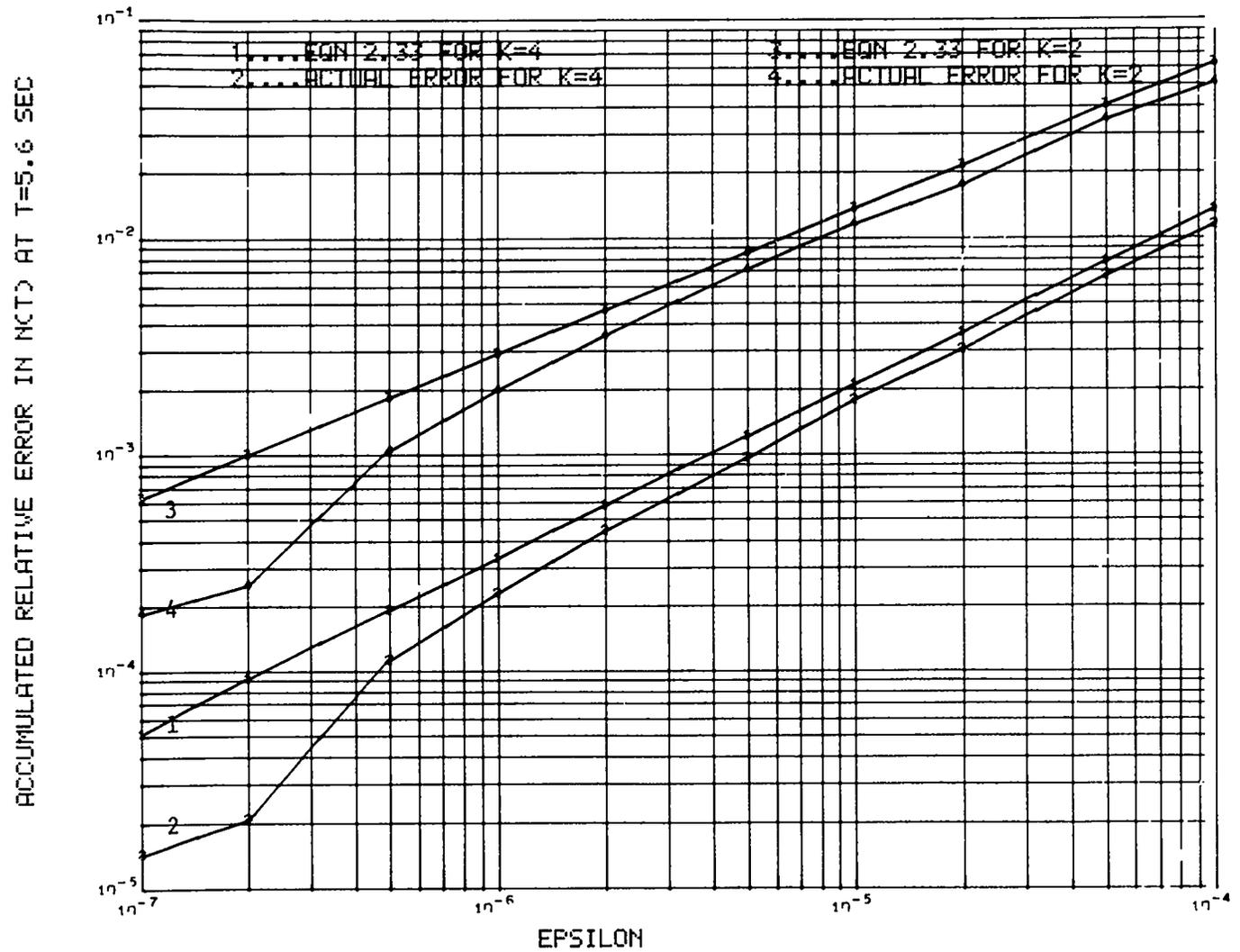
FIG 3.3G   ACCUMULATED RELATIVE ERROR FOR N(T) AS A FUNCTION OF EPSILON
WITH K AS PARAMETER, COMPUTED AT T=5.6 SEC FOR CASE 3.

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Approximate results using continuous analytic continuation with K = 4 and $\epsilon = 10^{-6}$ are compared with analytic results in Figs. 3.4A through 3.4C. Two cycles of the solution are plotted for the case $\Lambda = 10^{-5}$ sec, $\rho_0 = 5 \times 10^{-4}$, and $\omega = 10$ sec^{-1}. These figures show that the approximate results are in excellent agreement with the analytic results. For this problem, the odd derivatives of N(t) at t = 0 are zero, so that the initial value of the first time step cannot be computed from Eq. 2.10 when K is even. The results in Figs. 3.4A through 3.4C indicate, as in Case 2, that the procedure (Section 2.3.2) used to eliminate this difficulty is satisfactory.

The actual accumulated relative error in N(t) for this case is compared with Eq. 2.33 in Fig. 3.4D, and the results are in agreement with Eq. 2.13. Because of the oscillatory behavior of N(t) in this problem, the remainder in the Taylor expansion for N(t) changes sign over portions of the transient. As a result, there is a periodic cancellation of truncation errors, and the error does not accumulate as quickly as for nonoscillatory transients (compare Fig. 3.4D with Figs. 3.2E and 3.1C).

Figure 3.4E shows, as a function of ϵ, the actual relative error in N(t) and the predicted upper limit from Eq. 2.33 at t = 1 sec for two values of K. The results are in agreement with Eq. 2.13 and with the observation (Section 2.4.2) that, at a fixed point in the transient, the relative error is proportional to $\epsilon^{K/K+1}$.

The computing time for this problem was only about 0.03 min by continuous analytic continuation. However, by continuing the solution over many (≈ 50) cycles, computing times of the order of one minute can be obtained. Figure 3.4F shows the variation of the efficiency with ϵ for two values of K for the case shown in Figs. 3.4A through 3.4C. Results in Fig. 3.4F are in agreement with those in Section 2.4.2 (Fig. 2.3). Figure 3.4G shows the efficiency as a function of K with ϵ as parameter

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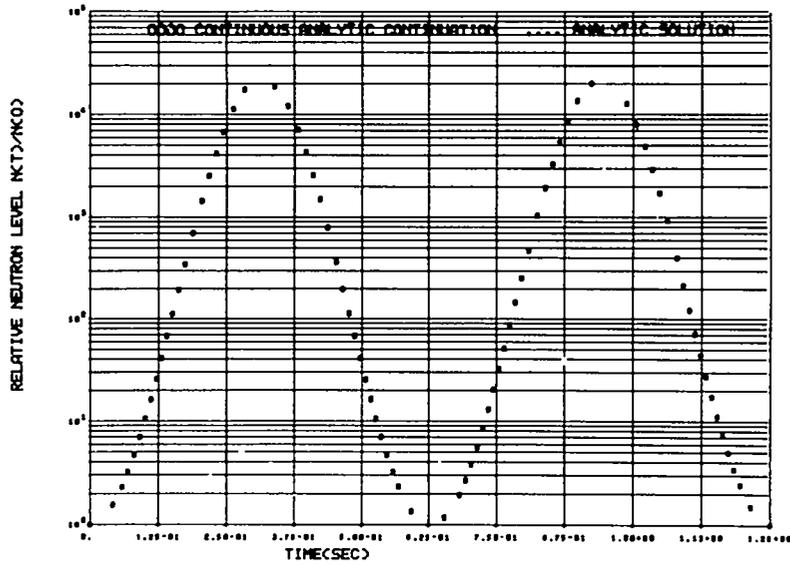
FIG 3.4A COMPARISON OF APPROXIMATE AND ANALYTIC RESULTS FOR CASE 4.
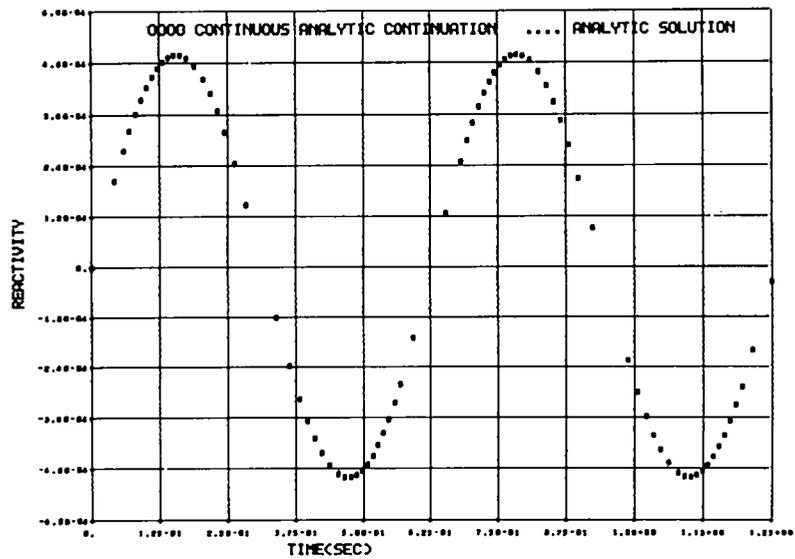

FIG 3.4B COMPARISON OF APPROXIMATE AND ANALYTIC RESULTS FOR CASE 4.
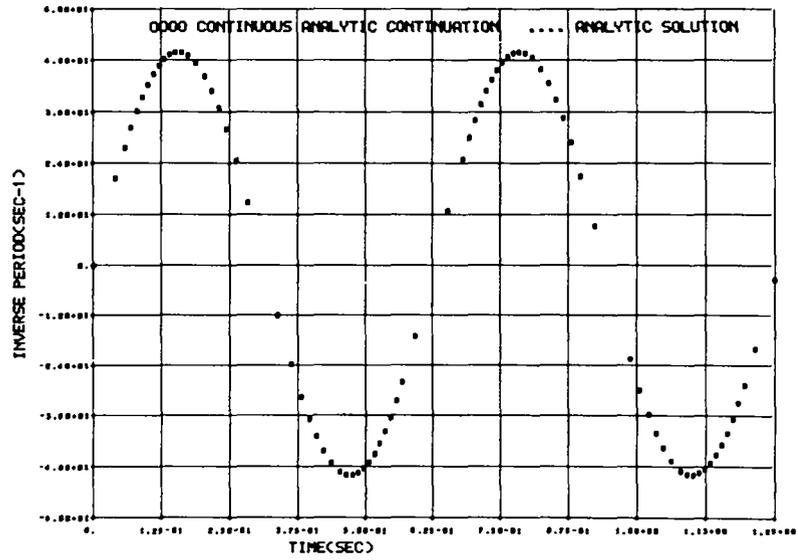
FIG 3.4C COMPARISON OF APPROXIMATE AND ANALYTIC RESULTS FOR CASE 4.
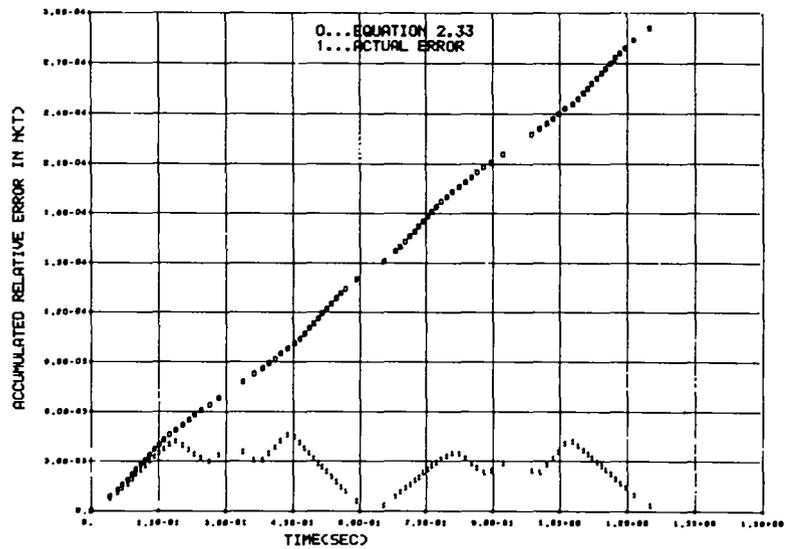

FIG 3.4D COMPARISON OF ACTUAL RELATIVE ERROR WITH EQN 2.33 FOR CASE 4.

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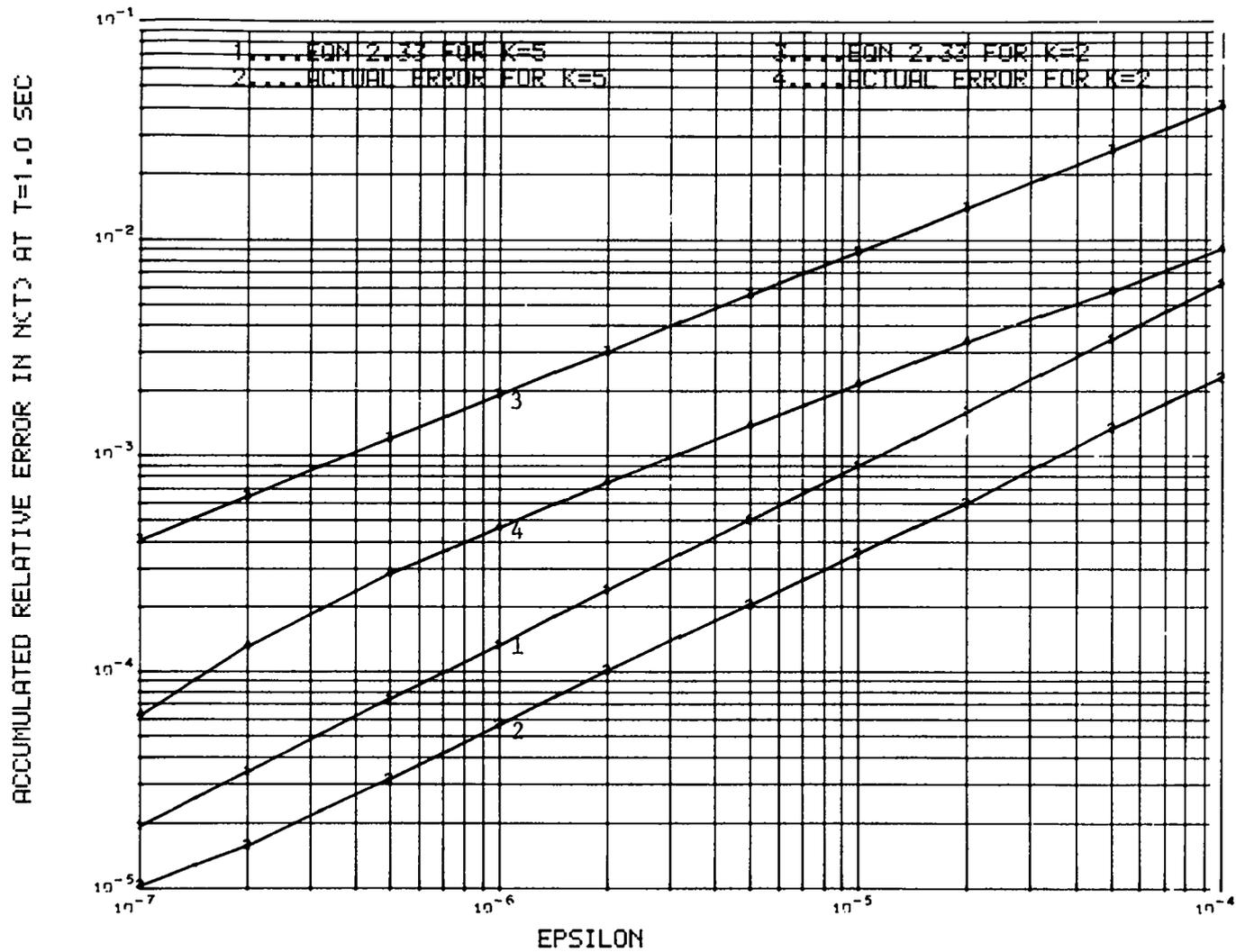
FIG 3.4E   ACCUMULATED RELATIVE ERROR FOR N(T) AS A FUNCTION OF EPSILON
WITH K AS PARAMETER, COMPUTED AT T=1.0 SEC FOR CASE 4.
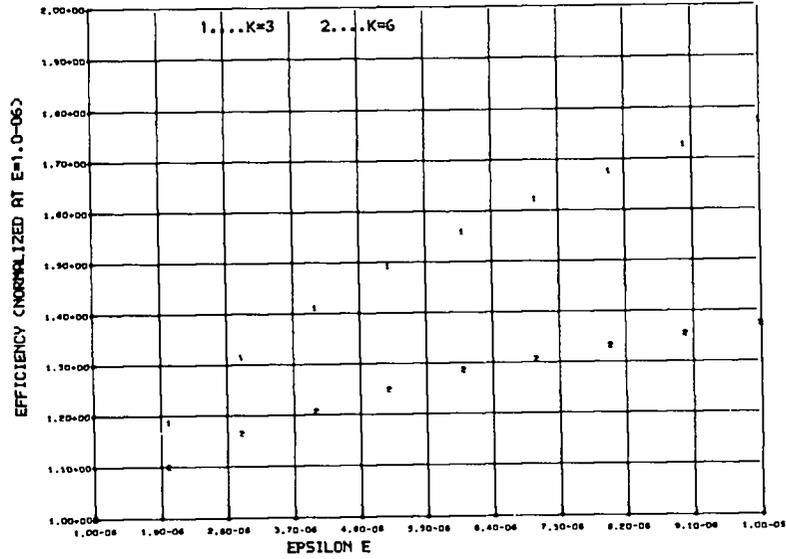
FIG 3.4F EFFICIENCY VERSUS EPSILON WITH K AS PARAMETER FOR CASE 4.
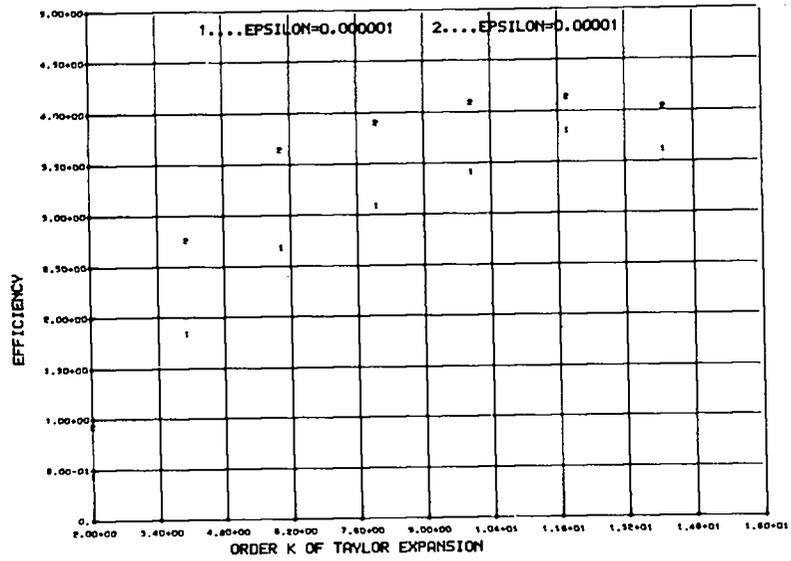
FIG 3.4G EFFICIENCY VERSUS K WITH EPSILON AS PARAMETER FOR CASE 4.

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for the case $\Lambda = 10^{-3}$ sec, $\rho_o = 10^{-2}$, and $\omega = 10$ sec^{-1}. The results in Fig. 3.4G are in qualitative agreement with those in Section 2.4.2 (Fig. 2.1).

3.1.5. Case 5: Step Change in Reactivity with One Delayed Neutron Group; No Source or Feedback

The problem of a step change in reactivity with one delayed neutron group and no source or feedback can be solved analytically. This problem is described by the equations

$$N^{(1)}(t) = \Lambda^{-1} (\rho - \beta) N(t) + \lambda C(t) \tag{3.26}$$

$$C^{(1)}(t) = \Lambda^{-1} \beta N(t) - \lambda C(t) \tag{3.27}$$

where ρ is a constant. The analytic solutions of Eqs. 3.26 and 3.27, obtained by Laplace transformation, are

$$\frac{N(t)}{N(0)} = a_1 e^{\omega_1 t} + a_2 e^{\omega_2 t} \tag{3.28}$$

$$\frac{C(t)}{C(0)} = b_1 e^{\omega_1 t} + b_2 e^{\omega_2 t} \tag{3.29}$$

$$\int_o^t N(t') \, dt' = N(0) \left[\frac{a_1}{\omega_1} (e^{\omega_1 t} - 1) + \frac{a_2}{\omega_2} (e^{\omega_2 t} - 1) \right] \tag{3.30}$$

where

$$a_1 = (\omega_1 + \lambda + \Lambda^{-1} \beta)/(\omega_1 - \omega_2)$$

$$a_2 = (\omega_2 + \lambda + \Lambda^{-1} \beta)/(\omega_2 - \omega_1)$$

$$b_1 = [\omega_1 + \lambda + \Lambda^{-1}(\beta - \rho)]/(\omega_1 - \omega_2)$$

$$b_2 = [\omega_2 + \lambda + \Lambda^{-1}(\beta - \rho)]/(\omega_2 - \omega_1).$$

In the above equations, ω_1 and ω_2 are the two roots of the equation

$$\omega^2 + [\lambda - \Lambda^{-1}(\rho - \beta)]\,\omega - \Lambda^{-1}\,\lambda\,\rho = 0$$

and the initial condition $C^{(1)}(0) = 0$ has been applied.

A comparison of approximate results using continuous analytic continuation with $K = 5$ and $\varepsilon = 10^{-6}$ is made with the analytic results in Figs. 3.5A through 3.5C. The results are for the case $\Lambda = 10^{-5}$ sec, $\rho = 3.2 \times 10^{-3}$, $\beta = 6.4 \times 10^{-3}$, and $\lambda = 0.4$ sec^{-1}. As can be seen from these figures, the approximate results are in excellent agreement with the analytic results.

Figure 3.5D compares the actual accumulated relative (or fractional) error in N(t) and C(t) with Eq. 2.33; the results are in agreement with Eq. 2.13. The actual relative error and the upper limit from Eq. 2.33 are plotted in Fig. 3.5E as a function of ε with K as parameter at $t = 25$ sec for the case under consideration. The results in Fig. 3.5E are in agreement with Eq. 2.13 and the accumulated relative error is proportional to $\varepsilon^{K/K+1}$ as expected (Section 2.4.2).

The computing time for this problem was not long enough (≈ 0.05 min) to allow investigation of the efficiency as a function of K and ε.

A straightforward application of the basic method to this case, and the case which follows, is not the most efficient from the viewpoint of computing time. Thus, a variation of the basic method was used in these cases (Section 2.5.1).
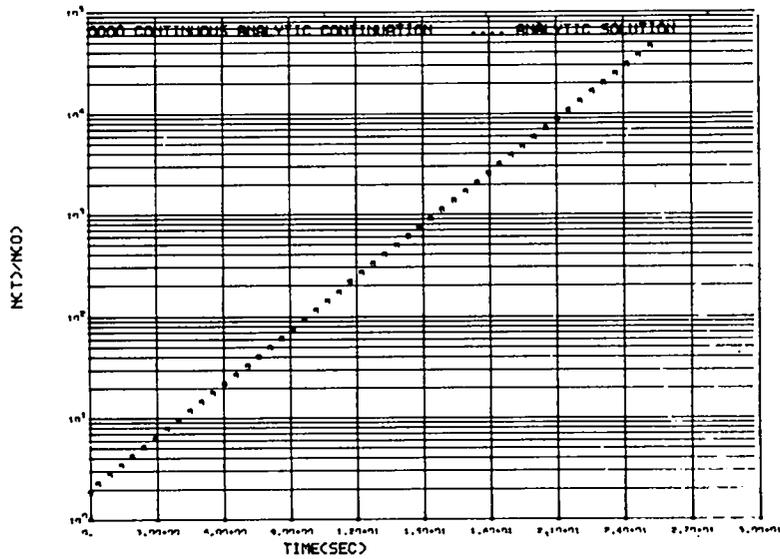
FIG 3.5A COMPARISON OF APPROXIMATE AND ANALYTIC RESULTS FOR CASE 5.
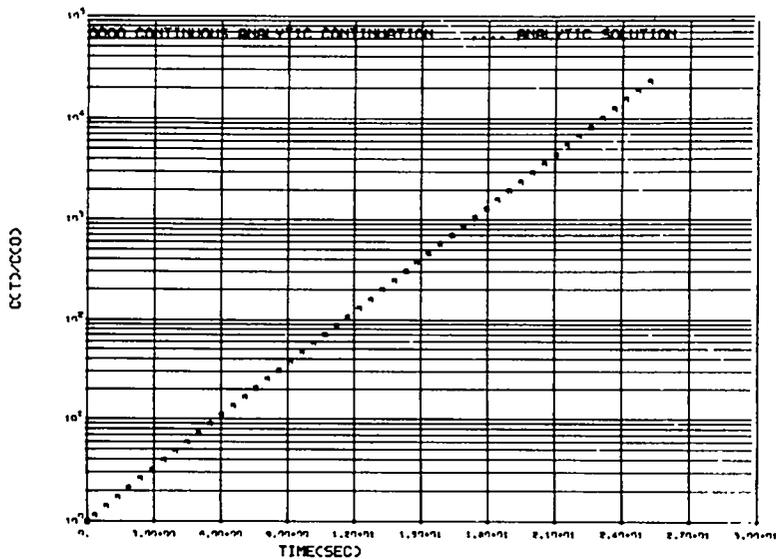

FIG 3.5B COMPARISON OF APPROXIMATE AND ANALYTIC RESULTS FOR CASE 5.
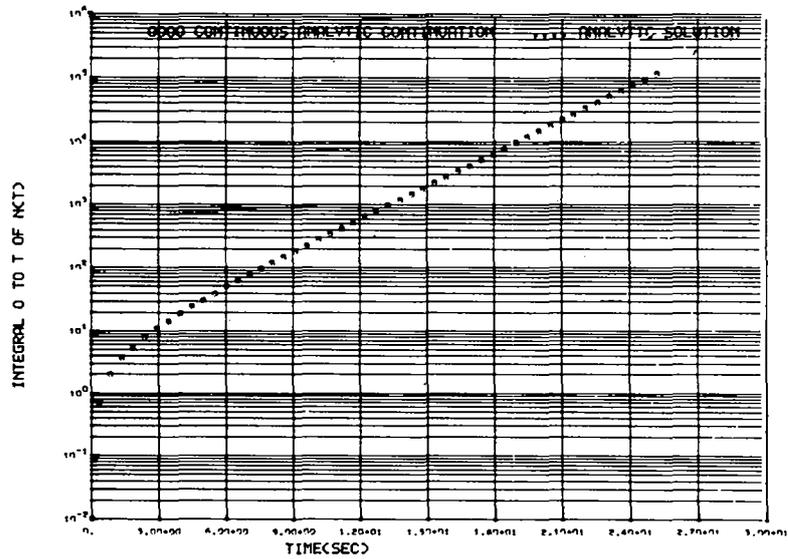
FIG 3.5C COMPARISON OF APPROXIMATE AND ANALYTIC RESULTS FOR CASE 5.
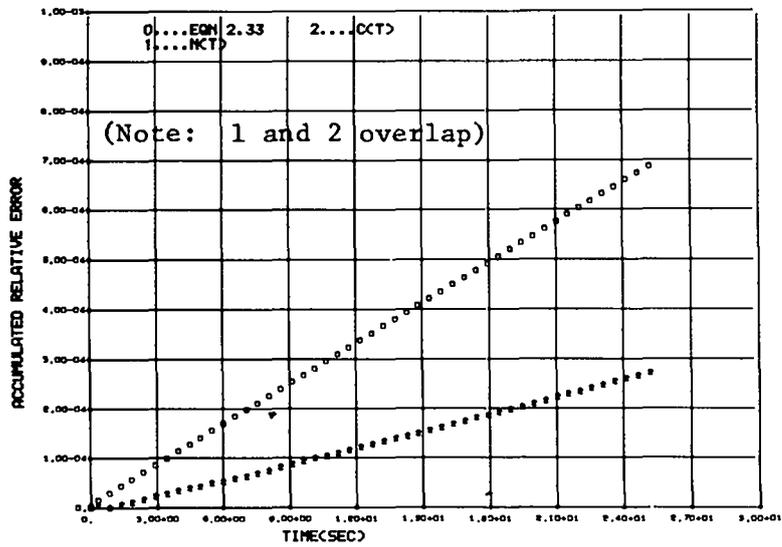

FIG 3.5D COMPARISON OF ACTUAL RELATIVE ERROR FOR
N(T) AND C(T) WITH EQN 2.33 FOR CASE 5.

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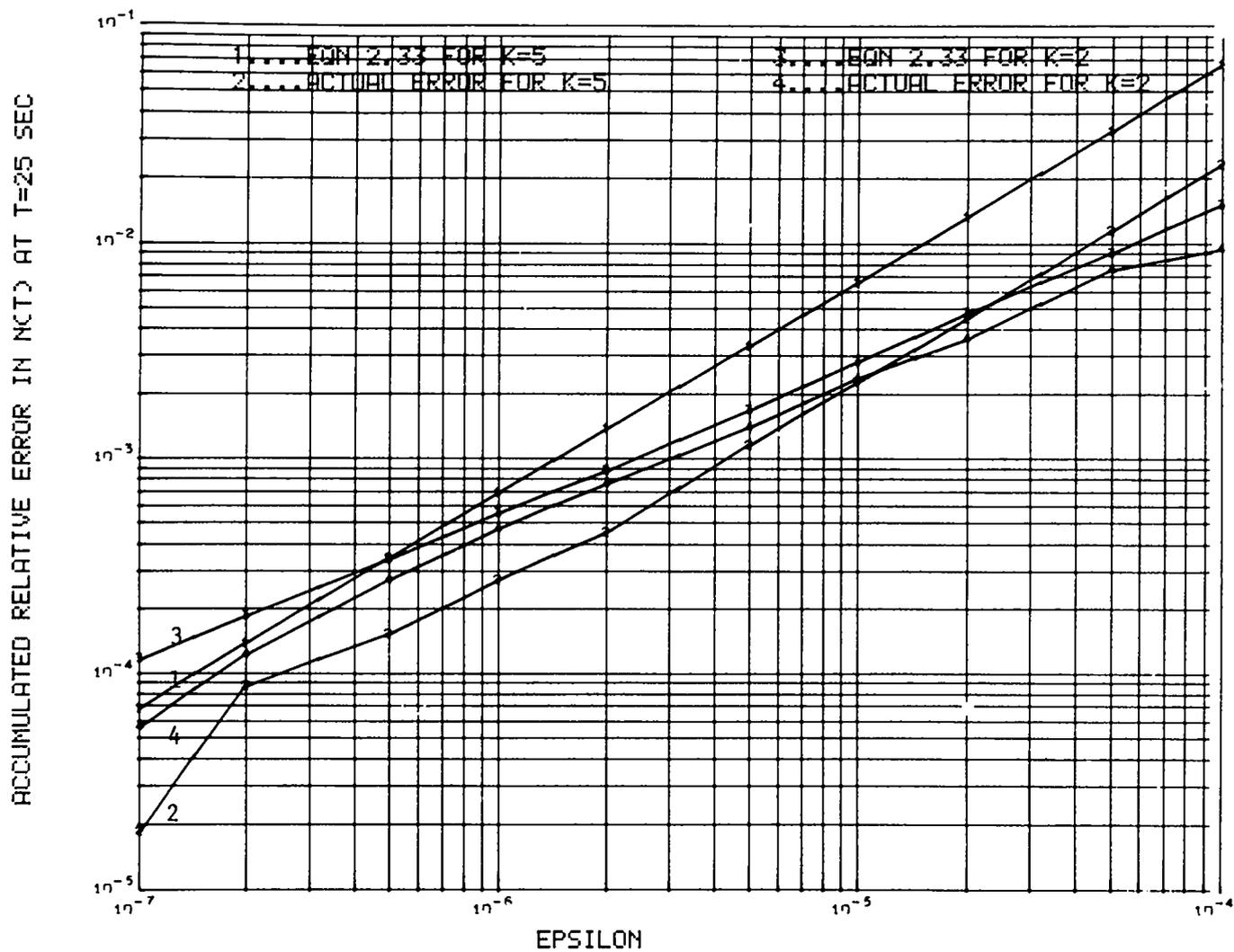
FIG 3.5E ACCUMULATED RELATIVE ERROR FOR N(T) AS A FUNCTION OF EPSILON
WITH K AS PARAMETER,COMPUTED AT T=25 SEC FOR CASE 5.

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3.1.6. Case 6: Step Change in Reactivity with Six Delayed Neutron Groups; No Source or Feedback

The final problem considered for which analytic solutions are available is that of a step input of reactivity with six delayed neutron groups and no source or feedback. This problem is an extension of Case 5 to more than one delayed neutron group. The equations to be solved are

$$N^{(1)}(t) = \Lambda^{-1} (\rho - \beta) N(t) + \sum_{i=1}^{6} \lambda_i C_i(t) \tag{3.31}$$

$$C_i^{(1)}(t) = \Lambda^{-1} \beta_i N(t) - \lambda_i C_i(t) \qquad i = 1, \ldots, 6 \tag{3.32}$$

where ρ is a constant.

By assuming the trial solutions

$$N(t) \; \alpha \; e^{\omega t}$$

$$C_i(t) \; \alpha \; e^{\omega t}$$

in Eqs. 3.31 and 3.32, it is found that

$$N(t) = \sum_{j=1}^{7} a_j \, e^{\omega_j t} \tag{3.33}$$

$$C_i(t) = \sum_{j=1}^{7} b_j^i \, e^{\omega_j t} \tag{3.34}$$

where the coefficients in Eqs. 3.33 and 3.34 are constants and the ω_j are the seven roots of the well-known inhour equation:

$$\rho = \Lambda \omega + \sum_{i=1}^{6} \beta_i [1 + (\lambda_i/\omega)]. \tag{3.35}$$

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In terms of the period $T = 1/\omega$, Eq. 3.35 can be written in the more familiar form

$$\rho = \frac{\Lambda}{T} + \sum_{i=1}^{6} \frac{\beta_i}{1 + \lambda_i T} \, . \tag{3.36}$$

Equations 3.31 and 3.32 were solved for step inputs of reactivity using continuous analytic continuation with $K = 4$ and $\epsilon = 10^{-6}$. The asymptotic behavior of $N(t)$ and $C_i(t)$ is proportional to $e^{(t/T)}$ where T is the largest root of Eq. 3.36. Asymptotic periods obtained with the approximate method are compared in Table I with those calculated from Eq. 3.36. The agreement between approximate and analytic results is excellent and the fractional error in the periods are within the limits predicted by the approximate method.

It should be pointed out again that Cases 5 and 6 were computed using a variation of the basic method (Section 2.5.1) which is applicable to these cases and which is more efficient than a straightforward application of the basic method.

TABLE I

Asymptotic Periods for Step Inputs of Reactivity
With Six Delayed Neutron Groups

Inhour Equation	Continuous Analytic Continuation, K=4, ϵ=10^{-6}	% Error
1.00000×10^0 sec	0.99999×10^0 sec	1.00×10^{-3}
1.00000×10^1	1.00000×10^1	$<10^{-3}$
2.00000×10^1	2.00000×10^1	$<10^{-3}$
3.00000×10^1	3.00001×10^1	-0.33×10^{-3}
4.00000×10^1	4.00002×10^1	-0.50×10^{-3}
5.00000×10^1	4.99998×10^1	0.40×10^{-3}
6.00000×10^1	5.99999×10^1	0.17×10^{-3}
7.00000×10^1	7.00001×10^1	-0.14×10^{-3}
8.00000×10^1	8.00000×10^1	$<10^{-3}$
9.00000×10^1	9.00001×10^1	-0.11×10^{-3}
1.00000×10^2	1.00000×10^2	$<10^{-3}$

3.2 Comparisons with a Numerical Integration Method

In the previous section, approximate results using continuous analytic continuation were compared with analytic solutions for a few cases, in order to check the accuracy of the approximate method and in order to verify some of its properties. In this Section, approximate solutions to two problems for which closed analytic solutions are not known are compared with approximate solutions from the RTS[7] code. This code solves a reduced integral form of the reactor kinetics equations by numerical integration using Simpson's rule.

3.2.1. Case 7: Ramp Input of Reactivity Starting at Source Equilibrium with Six Delayed Neutron Groups and Feedback Proportional to Total Energy Release

The first problem considered is that of a ramp input of reactivity starting at source equilibrium with six delayed neutron groups and reactivity feedback proportional to the integrated power or to the total energy release. This problem is represented by the equations

$$N^{(1)}(t) = \Lambda^{-1} [\rho(t) - \beta] N(t) + \sum_{i=1}^{6} \lambda_i C_i(t) + S \qquad (3.37)$$

$$C_i^{(1)}(t) = \Lambda^{-1} \beta_i N(t) - \lambda_i C_i(t) \qquad i = 1, \ldots, 6 \qquad (3.38)$$

$$\rho(t) = \rho_0 + at + b \int_0^t N(t') \, dt'. \qquad (3.39)$$

Equation 3.39 is equivalent to the two equations

$$\rho(t) = \rho_0 + at + \alpha[T(t) - T(0)] \qquad (3.40)$$

$$T^{(1)}(t) = HN(t) \qquad (3.41)$$

if $b = \alpha H$.

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Equations 3.37 through 3.39 were solved using continuous analytic continuation (ANCON code, Appendix D) and the RTS[7] code for the case

$$\Lambda = 10^{-8} \text{ sec} \qquad S = 3.61 \times 10^{4} \text{ neuts/sec} \qquad \rho_o = -2.1 \times 10^{-3}$$
$$a = 2.1 \times 10^{-1} \text{ sec}^{-1} \qquad b = -2.216 \times 10^{-8} \text{ sec}^{-1} \qquad \beta = 2.1 \times 10^{-3}$$
$$\beta_1 = 7.98 \times 10^{-5} \qquad \beta_2 = 5.88 \times 10^{-4} \qquad \beta_3 = 4.536 \times 10^{-4}$$
$$\beta_4 = 6.888 \times 10^{-4} \qquad \beta_5 = 2.163 \times 10^{-4} \qquad \beta_6 = 7.35 \times 10^{-5}$$
$$\lambda_1 = 1.29 \times 10^{-2} \text{ sec}^{-1} \qquad \lambda_2 = 3.11 \times 10^{-2} \qquad \lambda_3 = 1.34 \times 10^{-1}$$
$$\lambda_4 = 3.31 \times 10^{-1} \qquad \lambda_5 = 1.26 \qquad \lambda_6 = 3.21$$

Results using continuous analytic continuation with $K = 4$ and $\epsilon = 10^{-6}$ are compared with results from the RTS code in Figs. 3.6A through 3.6D. It can be seen from these figures that the agreement between the two results is satisfactory.

The computing time required by the RTS code for this excursion was 9.03 minutes, while only 0.35 minute was required by the ANCON code. The number of time steps required with the RTS code was 118,000 for an average time step of 2.56×10^{-7} sec, whereas with the ANCON code only 2,000 time steps were required for an average time step of 1.51×10^{-5} sec. Both problems were run on the same computer (IBM-7094) and the two codes are programmed in the same language (FORTRAN II, Version 3).

It was shown in Section 2.3 that by requiring that the relative truncation error in the Taylor expansion be at most equal to ϵ, an analytic criterion for the time step at each iteration could be obtained. In the RTS code, the starting time step is varied automatically through an arbitrary device. The criterion used in the RTS code to determine if the time step is to be modified is the magnitude of the relative change in $N(t)$ per time interval $|\delta N/N|$. The test sequence is as follows: If $|\delta N/N| < F_o$, Δt is set equal to the previous Δt. If $F_o \leq |\delta N/N| < F_1$, Δt is increased
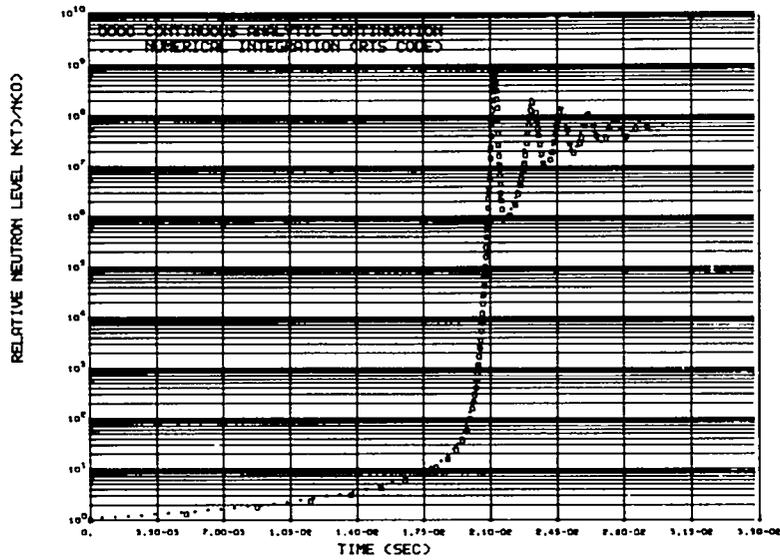
FIG 3.6A COMPARISON OF RESULTS FOR CASE 7 USING NUMERICAL INTEGRATION
(RTS CODE) AND CONTINUOUS ANALYTIC CONTINUATION.
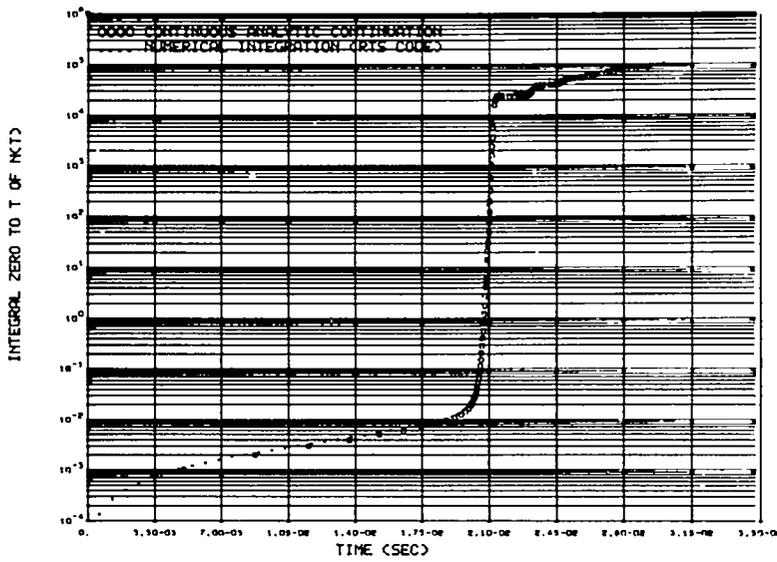
FIG 3.6B COMPARISON OF RESULTS FOR CASE 7 USING NUMERICAL INTEGRATION
(RTS CODE) AND CONTINUOUS ANALYTIC CONTINUATION.

64
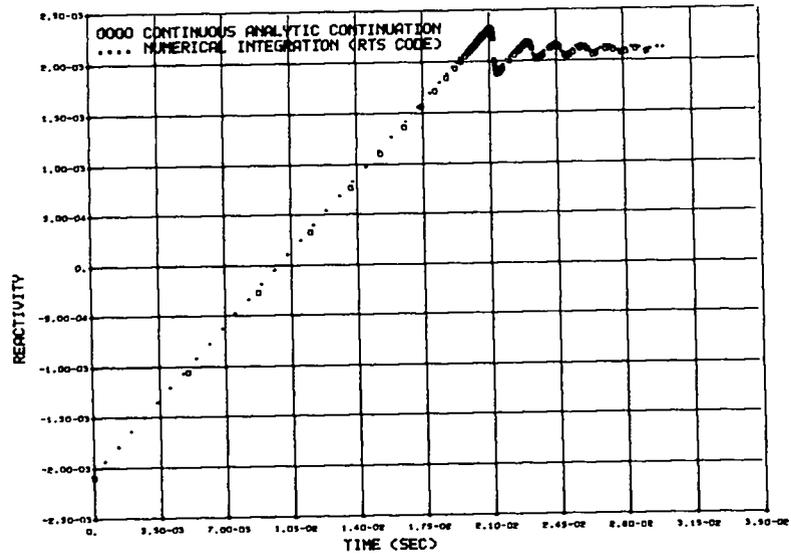
FIG 3.6C COMPARISON OF RESULTS FOR CASE 7 USING NUMERICAL INTEGRATION
(RTS CODE) AND CONTINUOUS ANALYTIC CONTINUATION.
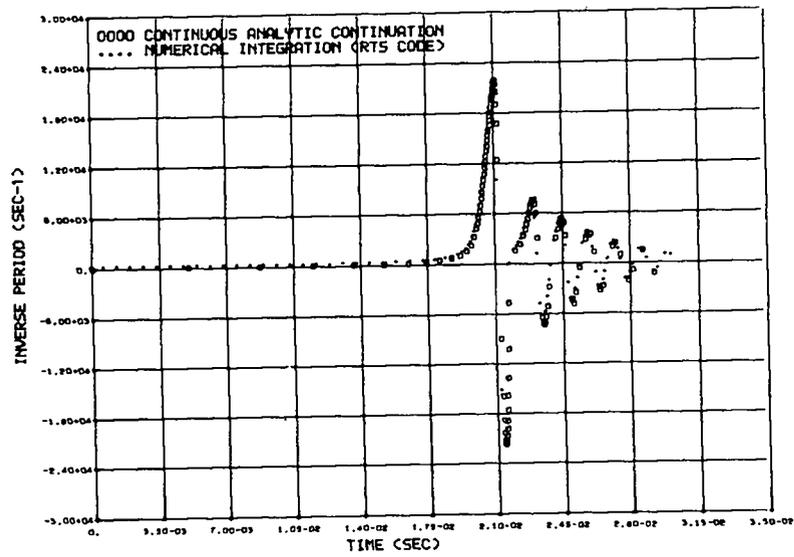
FIG 3.6D COMPARISON OF RESULTS FOR CASE 7 USING NUMERICAL INTEGRATION
(RTS CODE) AND CONTINUOUS ANALYTIC CONTINUATION.

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by a constant factor DT MULT. If $F_1 \leq |\delta N/N| \leq F_2$, Δt is unchanged. Finally, if $|\delta N/N| \geq F_2$, Δt is decreased by the factor DT MULT. Regardless of the results of the above tests, however, the minimum value of Δt allowed by the RTS code is 1×10^{-7} sec. In the above discussion, F_o, F_1, F_2, and DT MULT are input constants. The function of the F_o fiducial is to minimize spurious oscillations in $N(t)$ and $\omega(t)$ (inverse instantaneous period) which may develop when $|\delta N/N|$ becomes very small.

In the problem under consideration, the value used for DT MULT was 1.2, which is the value recommended in Ref. 7. However, the use of the recommended values for F_o, F_1, and F_2 (1×10^{-4}, 3×10^{-4}, and 3×10^{-3}, respectively) resulted in spurious oscillations in $\omega(t)$. Hence, these values were decreased until all spurious oscillations in $\omega(t)$ were eliminated. The final values used were $F_o = 1 \times 10^{-5}$, $F_1 = 3 \times 10^{-5}$, and $F_2 = 3 \times 10^{-4}$.

Figure 3.6E shows how the time step varies throughout the excursion for both methods. This figure should be examined in conjunction with Fig. 3.6A, since the time step is determined by the behavior of $N(t)$.

Figure 3.6F shows the upper limit of the relative or fractional error in $N(t)$ predicted by continuous analytic continuation (Eq. 2.33). Since $N(t)$ exhibits some damped oscillations (Fig. 3.6A), the actual fractional error in $N(t)$ using continuous analytic continuation can be expected (Section 3.1) to lie well below the curve in Fig. 3.6F. The error in the results from the RTS code cannot be predicted by the method used in that code. However, judging from Figs. 3.6A through 3.6D, the error is of the same order of magnitude as that using continuous analytic continuation.

The efficiency (Section 2.4), computed at $t = 3 \times 10^{-2}$ sec, is shown in Fig. 3.6G as a function of K for two values of ϵ. From this figure, it can be seen that, for $\epsilon = 10^{-6}$, $K = 5$ is the optimum order. The qualitative behavior of the efficiency as a function of K is as expected from the discussion in Section 2.4 (Fig. 2.2). Figure 3.6H shows the
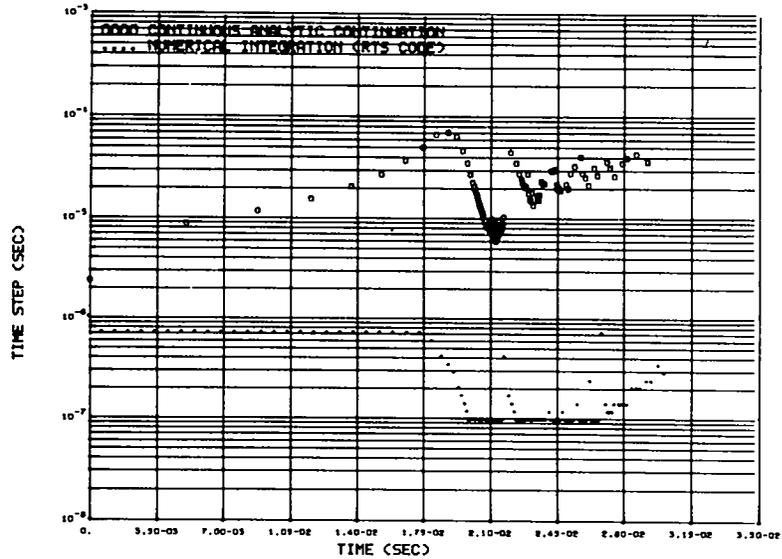
FIG 3.6E   COMPARISON OF RESULTS FOR CASE 7 USING NUMERICAL INTEGRATION
(RTS CODE) AND CONTINUOUS ANALYTIC CONTINUATION.
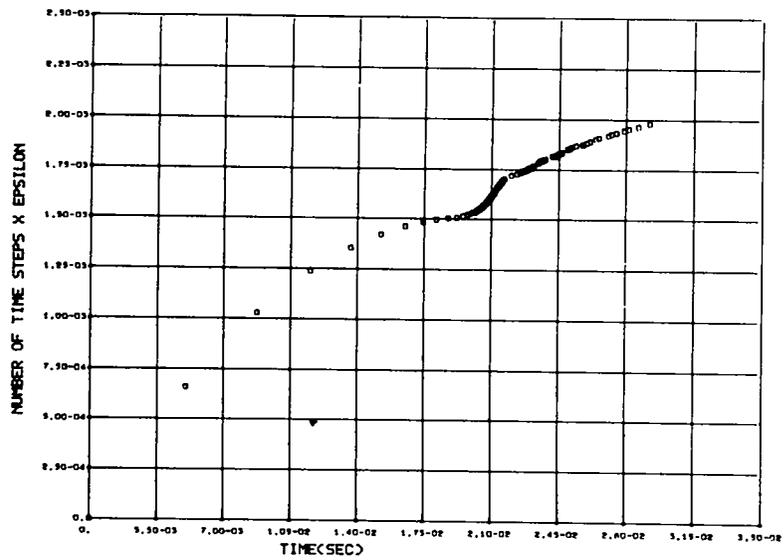
FIG 3.6F   UPPER LIMIT OF ACCUMULATED RELATIVE ERROR IN
N(T) FROM EQN 2.33 FOR CASE 7.

67
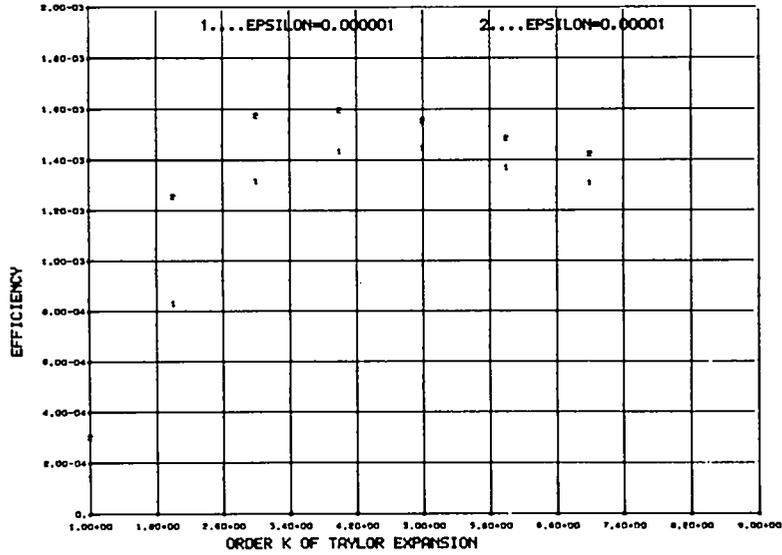
FIG 3.6G EFFICIENCY VERSUS K WITH EPSILON AS PARAMETER FOR CASE 7.
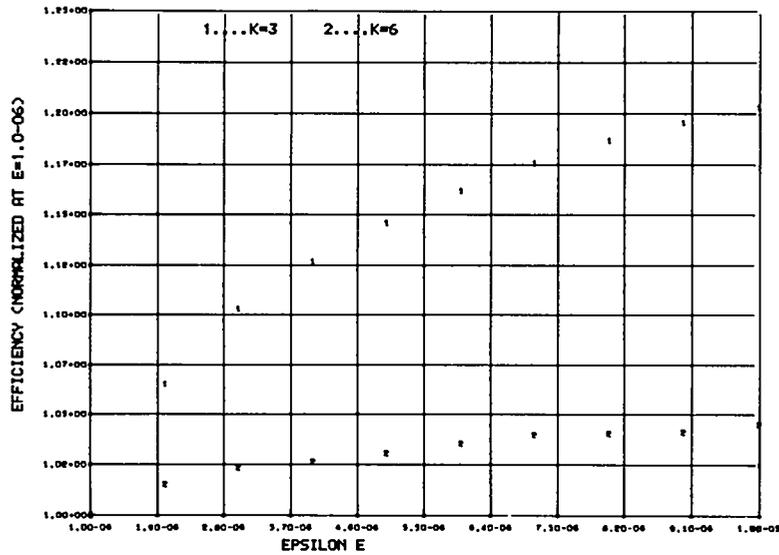
FIG 3.6H EFFICIENCY VERSUS EPSILON WITH K AS PARAMETER FOR CASE 7.

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efficiency (normalized at $\epsilon = 10^{-6}$) as a function of ϵ with K as parameter. The efficiency, and hence the computing time, is not strongly dependent on the error criterion ϵ, especially for larger values of K. These results are in qualitative agreement with those in Section 2.4 (Fig. 2.3).

3.2.2. Case 8: Step Input of Reactivity with Six Delayed Neutron Groups and Feedback Proportional to Total Energy Release

The second problem considered is that of a step input of reactivity with six delayed neutron groups and reactivity feedback proportional to the integrated power or total energy release. This problem is represented by equations similar to the previous case (Case 7) but with $S = 0$ and $a = 0$.

Figures 3.7A through 3.7D are a comparison of results using continuous analytic continuation and the RTS code for the case

$$\Lambda = 10^{-3} \text{ sec} \qquad \rho_o = 1 \times 10^{-2}$$
$$b = -1 \times 10^{-17} \text{ sec}^{-1} \qquad \beta = 6.5 \times 10^{-3}$$
$$\beta_1 = 2.145 \times 10^{-4} \qquad \beta_2 = 1.4235 \times 10^{-3} \qquad \beta_3 = 1.274 \times 10^{-3}$$
$$\beta_4 = 2.5675 \times 10^{-3} \qquad \beta_5 = 7.475 \times 10^{-4} \qquad \beta_6 = 2.73 \times 10^{-4}$$
$$\lambda_1 = 1.24 \times 10^{-2} \qquad \lambda_2 = 3.05 \times 10^{-2} \qquad \lambda_3 = 1.11 \times 10^{-1}$$
$$\lambda_4 = 3.01 \times 10^{-1} \qquad \lambda_5 = 1.14 \qquad \lambda_6 = 3.01$$

The constants used in these calculations were $K = 4$ and $\epsilon = 10^{-6}$ with continuous analytic continuation, while the recommended (Ref. 7) values of DT MULT $= 1.2$, $F_o = 10^{-4}$, $F_1 = 3 \times 10^{-4}$, and $F_2 = 3 \times 10^{-3}$ were used in the RTS code. As can be seen from Figs. 3.7A through 3.7D, the agreement between the two methods is very good in general. The only exception is in the tail of the transient where the inverse period calculated by the RTS code contains several spurious oscillations. These cannot be seen in Fig. 3.7D because of the scale used but are evident in the output listing.
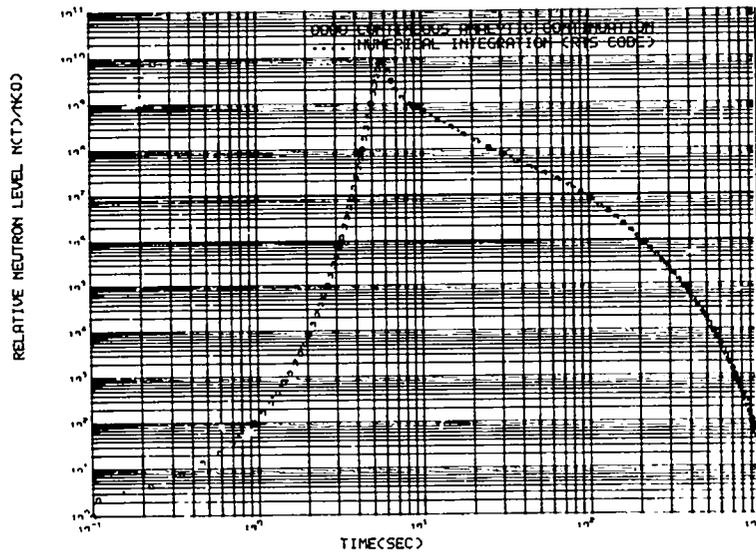
FIG 3.7A COMPARISON OF RESULTS FOR CASE 8 USING NUMERICAL INTEGRATION
(RTS CODE) AND CONTINUOUS ANALYTIC CONTINUATION.
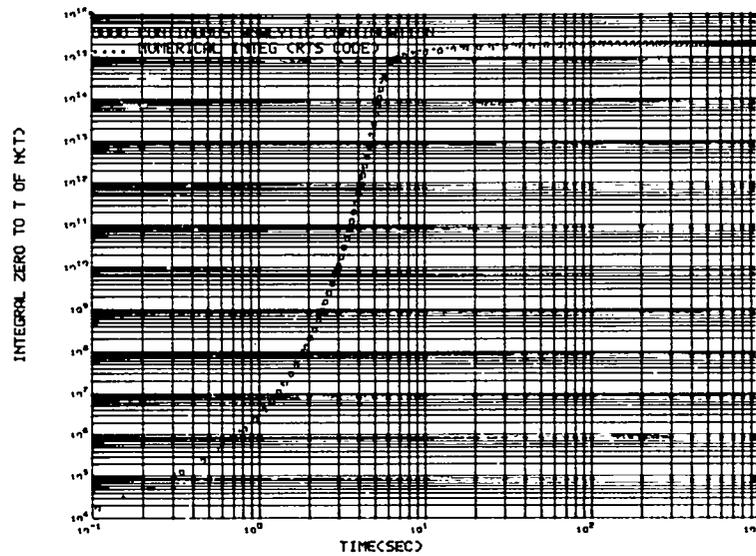
FIG 3.7B COMPARISON OF RESULTS FOR CASE 8 USING NUMERICAL INTEGRATION
(RTS CODE) AND CONTINUOUS ANALYTIC CONTINUATION.

70
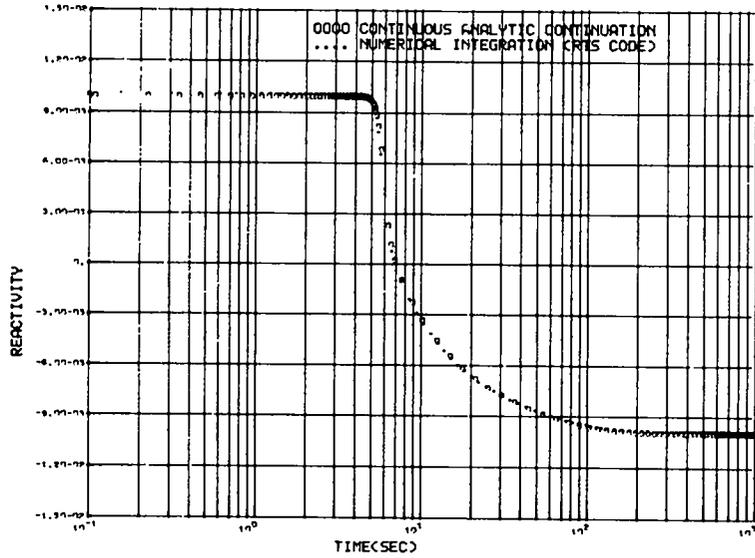
FIG 3.7C   COMPARISON OF RESULTS FOR CASE 8 USING NUMERICAL INTEGRATION
(RTS CODE) AND CONTINUOUS ANALYTIC CONTINUATION.
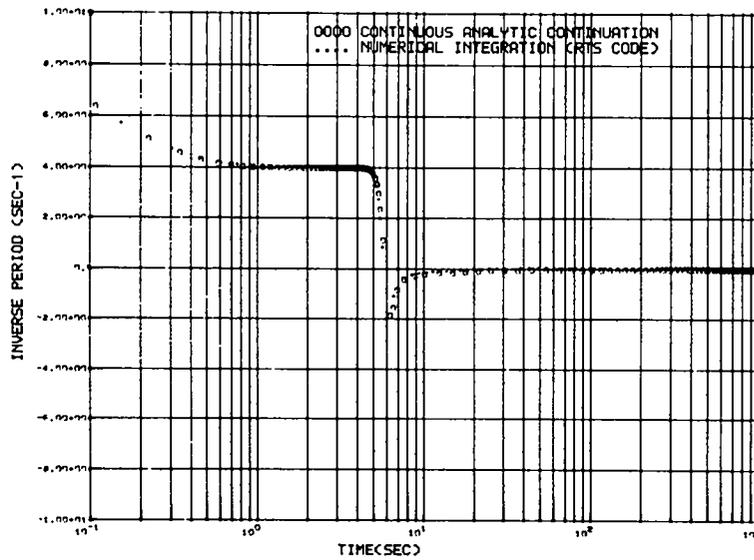
FIG 3.7D   COMPARISON OF RESULTS FOR CASE 8 USING NUMERICAL INTEGRATION
(RTS CODE) AND CONTINUOUS ANALYTIC CONTINUATION.

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The constants F_o, F_1, F_2 were decreased by an order of magnitude from the recommended values in an attempt to eliminate these spurious oscillations. However, this failed to eliminate the spurious oscillations, and the computing time increased by an order of magnitude.

With the ANCON code, 1.00 minute of computing time was required, while the RTS code required 2.99 minutes using the recommended values for F_o, F_1, and F_2. The number of time steps required to compute the excursion by continuous analytic continuation was 5,960 for an average time step of 1.68×10^{-1} sec. The corresponding numbers for the RTS code were 67,200 time steps and an average time step of 1.51×10^{-2} sec.

Figure 3.7E shows the variation of the time step as a function of time for both methods. The upper limit for the relative or fractional error predicted by continuous analytic continuation (Eq. 2.33) is plotted in Fig. 3.7F. From the previous work (for example, Case 3), the actual error in the results should lie well below the curve in Fig. 3.7F. The error in the results from the RTS code can be estimated from Figs. 3.7A through 3.7D to be of the same order of magnitude as that using continuous analytic continuation.

The variation in the efficiency with K and ε for this problem is shown in Figs. 3.7G and 3.7H. The results in these figures again verify some properties predicted in Section 2.4.2 for continuous analytic continuation.

It should be pointed out that the RTS code solves the kinetics equations based on the prompt neutron lifetime ℓ. These differ from those based on the prompt neutron generation time Λ in that the delayed neutron precursor equations based on ℓ are nonlinear, while those based on Λ are linear (Appendix A). Also, in the RTS calculations in this section, ℓ was taken to be numerically equal to Λ and $\delta k(t) = k(t) - 1$ was taken to be numerically equal to $\rho(t) = [k(t) - 1]/k(t)$. These differences are in addition to the difference in the approximate methods used to solve the equations. In spite of all these differences, the results using continuous analytic continuation are in good agreement with those using the RTS code.
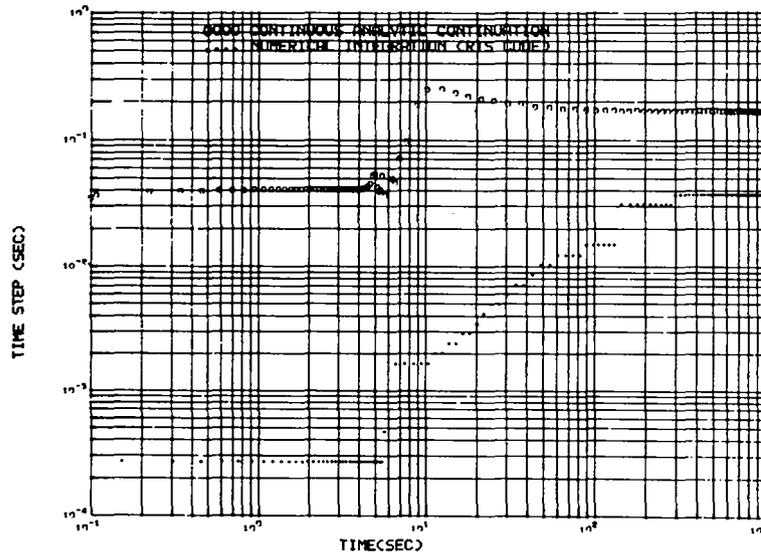
FIG 3.7E COMPARISON OF RESULTS FOR CASE 8 USING NUMERICAL INTEGRATION
(RTS CODE) AND CONTINUOUS ANALYTIC CONTINUATION.
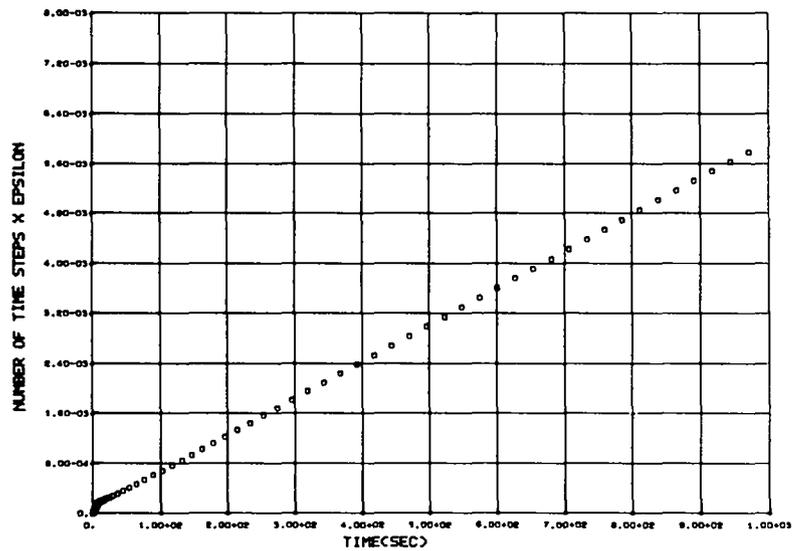
FIG 3.7F UPPER LIMIT OF ACCUMULATED RELATIVE ERROR IN
N(T) FROM EQN 2.33 FOR CASE 8.
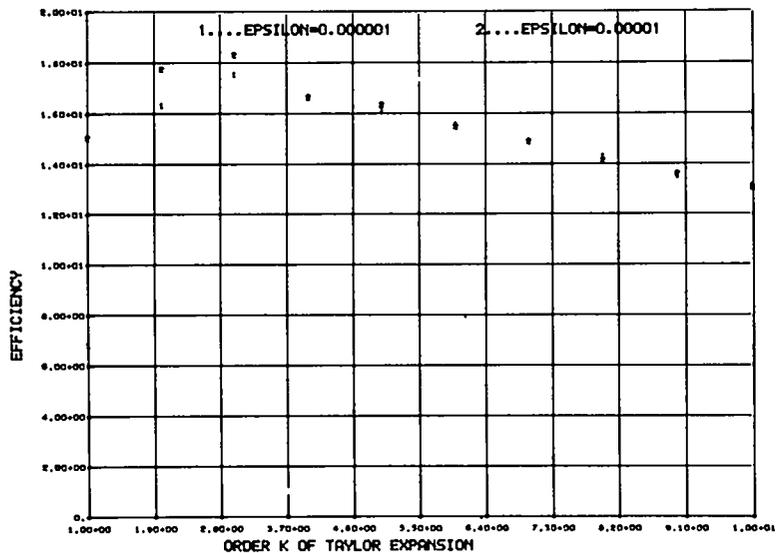
FIG 3.7G EFFICIENCY VERSUS K FOR CASE 8 (COMPUTED AT T=1000 SEC).
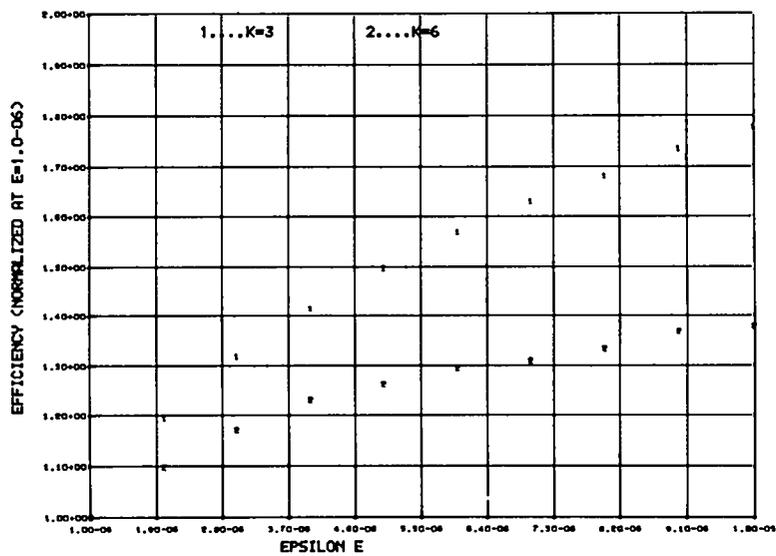
FIG 3.7H EFFICIENCY VERSUS EPSILON FOR CASE 8 (COMPUTED AT T=5.7 SEC).

This leads to the conclusion that it is immaterial which set of the reactor kinetics equations are used, and furthermore, ℓ and $\delta k(t)$ can be used interchangeably with Λ and $\rho(t)$, respectively, without significantly changing the results. These points are discussed further in Section 3.4.

3.3 Comparison with Experimental Transients

In this section, results obtained using continuous analytic continuation are compared with experimental excursions reported in the literature for the Godiva[22] and SPERT I[23,24] reactors. Godiva is a fast, bare U^{235} metal, spherical reactor with a diameter of about 6.75 inches. On the other hand, SPERT I is a reflected, water-moderated heterogeneous, low-enrichment UO_2 thermal reactor.

3.3.1. Godiva Bursts

Reference 22 reports the results of experimental studies made of the time behavior of Godiva following step increases in reactivity with essentially zero initial power level and no external neutron source. For step inputs of reactivity in the prompt critical range, but below about 5¢ above prompt critical, the reactivity feedback (which is due to thermal expansion of the metal sphere) is proportional to the total energy release. Thus, the equations which describe the time behavior of Godiva for step inputs of reactivity in this range are the same as those for Case 8 (Section 3.2.2). For slower transients (step inputs of reactivity less than prompt critical), the heat transfer from the sphere to its surroundings cannot be neglected. For faster transients (step inputs of reactivity larger than about 5¢ above prompt critical), the thermal expansion of the sphere is retarded by inertia.

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Assuming that $\beta = 6.4 \times 10^{-3}$, the neutronic parameters for Godiva[22] are

$\beta/\Lambda = 1.03 \times 10^{6}$ sec^{-1} $\quad\quad$ $\Lambda = 6.2136 \times 10^{-9}$ sec

$\lambda_1 = 1.2771 \times 10^{-2}$ sec^{-1} $\quad$ $\lambda_2 = 3.1949 \times 10^{-2}$ $\quad$ $\lambda_3 = 1.1820 \times 10^{-1}$

$\lambda_4 = 3.1847 \times 10^{-1}$ $\quad\quad\quad$ $\lambda_5 = 1.5083$ $\quad\quad\quad\quad$ $\lambda_6 = 5.3191$

$\beta_1 = 2.24 \times 10^{-4}$ $\quad\quad\quad\quad$ $\beta_2 = 1.3248 \times 10^{-3}$ $\quad$ $\beta_3 = 1.2224 \times 10^{-3}$

$\beta_4 = 2.6176 \times 10^{-3}$ $\quad\quad\quad$ $\beta_5 = 8.832 \times 10^{-4}$ $\quad$ $\beta_6 = 1.28 \times 10^{-4}$

temperature coefficient of reactivity = $-2.688 \times 10^{-5}/^{\circ}$C

heat capacity = 6.6613×10^{-3} MW-sec/$^{\circ}$C.

Calculated results using continuous analytic continuation are compared with experimental results in Figs. 3.8A through 3.8C. Figure 3.8A is a plot of the peak fission rate, which is proportional to the peak neutron level, versus the initial asymptotic inverse period ω_0. Figure 3.8B shows the burst yield or total fissions to the peak as a function of ω_0. The burst yield is proportional to the integrated neutron level at the peak. Figure 3.8C is a plot of the burst width at half-maximum as a function of ω_0. The calculated results are in good agreement with the experimental results in the region where the feedback model used is valid. Above $\omega_0 = 5 \times 10^{4}$ sec^{-1}, corresponding to a step input of approximately 5¢ above prompt critical, the retardation of the thermal expansion of the reactor by inertia becomes increasingly important.

3.3.2. SPERT I Excursions

Experimental results of excursions in SPERT I following step inputs of reactivity are reported in Ref. 23. For $\omega_0 > 100$ sec^{-1}, corresponding to step inputs >30¢ above prompt critical, heat transfer from the fuel to the moderator can be neglected and the reactivity[24] is given by

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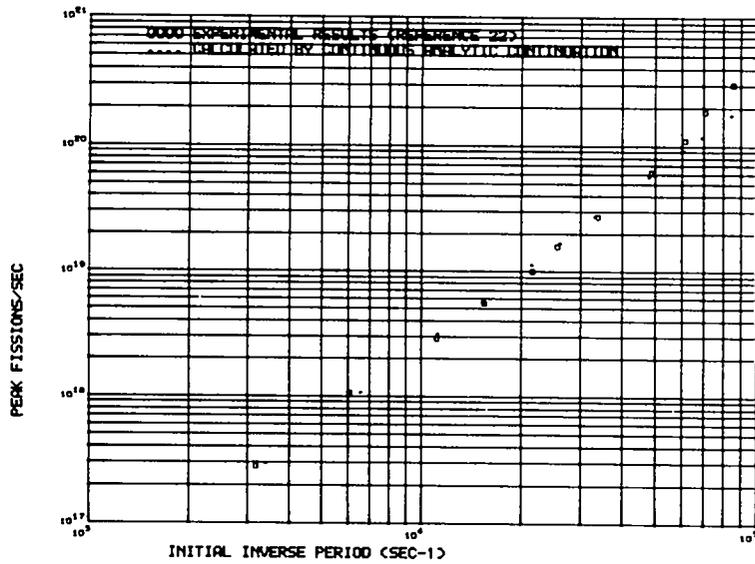
FIG 3.8A   COMPARISON OF CALCULATED AND EXPERIMENTAL PEAK FISSION RATE
AS A FUNCTION OF INITIAL INVERSE PERIOD FOR GODIVA REACTOR.
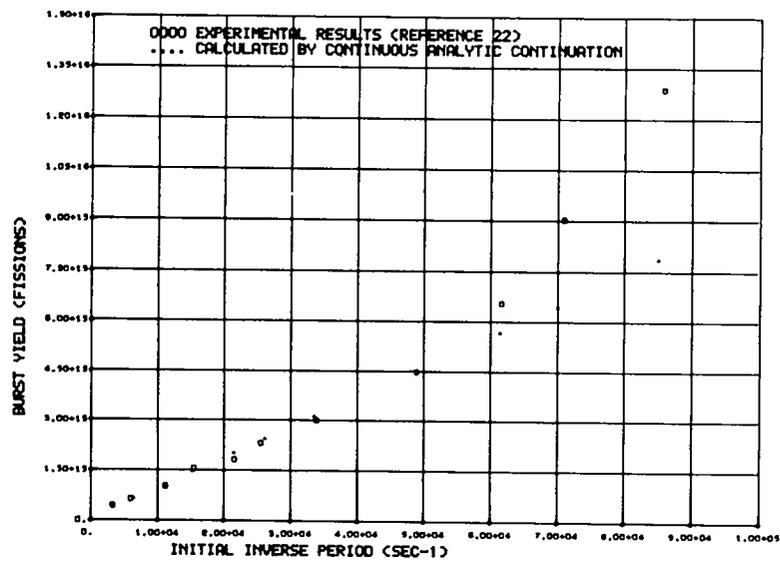
FIG 3.8B   COMPARISON OF CALCULATED AND EXPERIMENTAL FISSION YIELD AT
PEAK VERSUS INITIAL INVERSE PERIOD FOR GODIVA REACTOR.
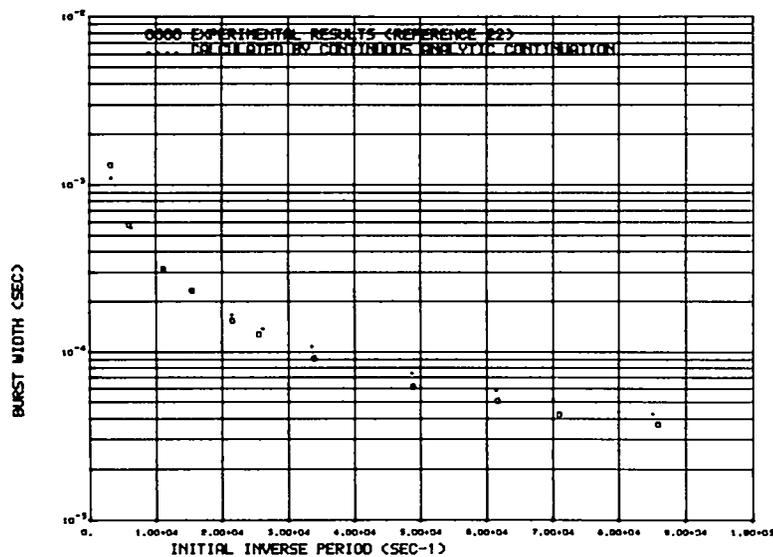
FIG 3.8C COMPARISON OF CALCULATED AND EXPERIMENTAL BURST WIDTH AT HALF
MAXIMUM VERSUS INITAL INVERSE PERIOD FOR GODIVA REACTOR.

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$$\rho(t) = \rho_0 - 4.588 \times 10^{-4} E^{0.74}$$

where E is the integrated power or total energy release. For $\omega_0 > 100$ sec^{-1}, reactivity feedback in the constrained (no fuel element bowing) SPERT I core is due mostly to Doppler broadening of U^{238} absorption resonances as the fuel temperature increases. Other neutronic parameters[23,24] for SPERT I are

$\Lambda/\beta = 3.6 \quad \times 10^{-3}$ sec $\qquad \beta = 7.4 \quad \times 10^{-3} \qquad \Lambda = 2.664 \quad \times 10^{-5}$ sec

$\lambda_1 = 1.27 \quad \times 10^{-2}$ sec$^{-1} \quad \lambda_2 = 3.17 \quad \times 10^{-2} \qquad \lambda_3 = 1.16 \quad \times 10^{-1}$

$\lambda_4 = 3.13 \quad \times 10^{-1} \qquad\qquad \lambda_5 = 1.40 \qquad\qquad\quad \lambda_6 = 3.88$

$\beta_1 = 2.4494 \times 10^{-4} \qquad\quad \beta_2 = 1.44744 \times 10^{-3} \quad \beta_3 = 1.24912 \times 10^{-3}$

$\beta_4 = 3.14352 \times 10^{-3} \qquad \beta_5 = 1.12924 \times 10^{-3} \quad \beta_6 = 1.8574 \times 10^{-4}$

heat capacity of active region $= 0.11(1 + 5.3 \times 10^{-4} \omega_0) \dfrac{\text{MW-sec}}{°\text{C}}$

Calculated results using continuous analytic continuation are compared with experimental results in Figs. 3.9A and 3.9B where the peak power and energy-to-peak power, respectively, are plotted as a function of ω_0. The agreement between calculated and experimental results is good.
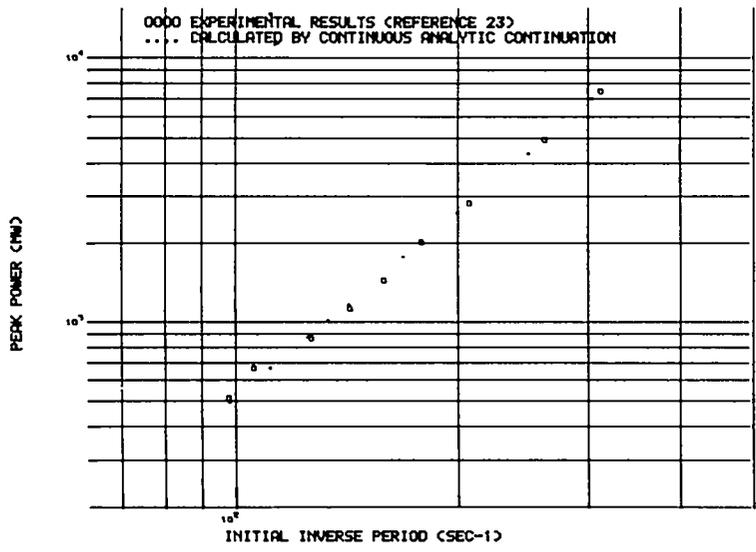
FIG 3.9A COMPARISON OF CALCULATED AND EXPERIMENTAL PEAK POWER AS A
FUNCTION OF INITIAL INVERSE PERIOD FOR SPERT I REACTOR.
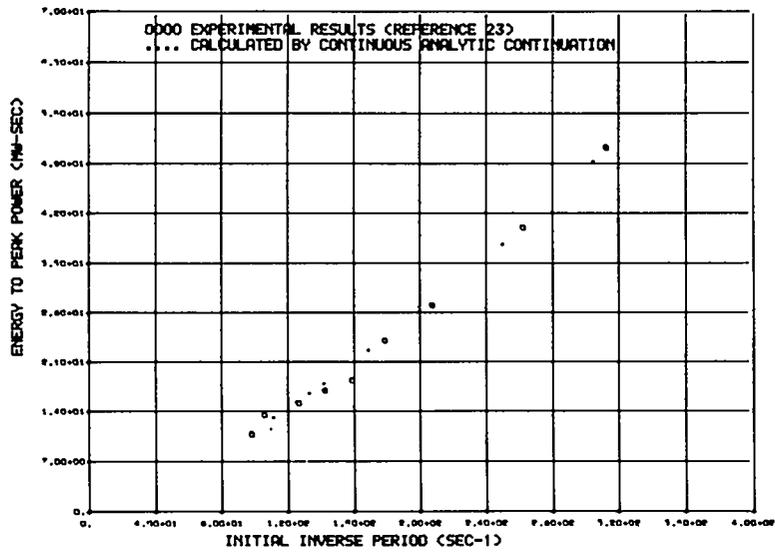
FIG 3.9B COMPARISON OF CALCULATED AND EXPERIMENTAL ENERGY TO PEAK AS A
FUNCTION OF INITIAL INVERSE PERIOD FOR SPERT I REACTOR.

3.4 Effect of Representation of Reactor Kinetics Equations

In Appendix A, it was shown that the reactor kinetics equations could assume two different forms; one form being based on the prompt neutron generation time Λ and the other on the prompt neutron lifetime ℓ. These two sets of equations are reproduced below for easy reference.

Equations based on Λ

$$N^{(1)}(t) = \Lambda^{-1}[\rho(t) - \beta] N(t) + \sum_{i=1}^{I} \lambda_i C_i(t) + S$$

$$C_i^{(1)}(t) = \Lambda^{-1} \beta_i N(t) - \lambda_i C_i(t) \qquad\qquad i = 1,\ldots,I$$

$$\rho(t) = [k(t) - 1]/k(t) \tag{3.42}$$

Equations based on ℓ

$$N^{(1)}(t) = \ell^{-1}[(1 - \beta) \delta k(t) - \beta] N(t) + \sum_{i=1}^{I} \lambda_i C_i(t) + S$$

$$C_i^{(1)}(t) = \ell^{-1} \beta_i [\delta k(t) + 1] N(t) - \lambda_i C_i(t) \qquad i=1,\ldots,I$$

$$\delta k(t) = k(t) - 1 \tag{3.43}$$

Again note that the basic difference between these two forms is in the equations for $C_i(t)$; those based on ℓ being nonlinear while those based on Λ are linear.

Some comparisons between continuous analytic continuation using the kinetics equations based on Λ and the RTS code, which solves a reduced form of the kinetics equations based on ℓ by numerical integration, were made in Section 3.2. The problems compared in that section were the response of a fast reactor to a ramp input of reactivity and of a thermal reactor to a step input of reactivity. For all practical purposes, the results from the two methods were identical.

In this section, the two forms of the reactor kinetics equations (Eqs. 3.42 and 3.43) are solved for a particular problem by continuous analytic continuation and the results compared. The problem used for comparison is that of a terminating ramp input of reactivity starting at source equilibrium with six delayed neutron groups and reactivity feedback proportional to the fuel and moderator temperatures. Heat transfer from the fuel to the moderator is assumed to take place by radiation. Thus, the equations which must be added to Eqs. 3.42 and 3.43 to complete the description are

$$\rho(t) = \rho_o + at + \alpha_f[T_f(t) - T_f(0)] + \alpha_m[T_m(t) - T_m(0)] \qquad t \leq t_1$$

$$\rho(t) = \rho_o + at_1 + \alpha_f[T_f(t) - T_f(0)] + \alpha_m[T_m(t) - T_m(0)] \qquad t > t_1$$

$$\tag{3.44}$$

$$c_f \, T_f^{(1)}(t) = Q_f \, HN(t) - h_r\left[T_f^4(t) - T_m^4(t)\right] \tag{3.45}$$

$$c_m \, T_m^{(1)}(t) = Q_m \, HN(t) + h_r\left[T_f^4(t) - T_m^4(t)\right] \tag{3.46}$$

in which t_1 is the time at which the ramp is terminated. Note that the heat balance equations (Eq. 3.45 and 3.46) are nonlinear, because of the T^4 terms. The equation which defines $\delta k(t)$ is the same as Eq. 3.44 with $\rho(t)$ and ρ_o replaced, respectively, by $\delta k(t)$ and δk_o.

Comparisons of results from the two forms of the kinetics equations using $K = 4$ and $\epsilon = 10^{-6}$ are made in Figs. 3.10A through 3.10D. The excursion shown in these figures is for the case

$\Lambda = 10^{-3}$ sec $\beta = 6.5 \quad \times \ 10^{-3}$ $S = 200$ neuts/sec

$\rho_o = \delta k_o = -2.5 \times 10^{-2}$ $a = 6.5 \times 10^{-4}$ sec^{-1} $t_1 = 47.5$ sec

$\lambda_1 = 1.24 \quad \times \ 10^{-2}$ sec^{-1} $\lambda_2 = 3.05 \quad \times \ 10^{-2}$ $\lambda_3 = 1.11 \quad \times \ 10^{-1}$

$\lambda_4 = 3.01 \quad \times \ 10^{-1}$ $\lambda_5 = 1.14$ $\lambda_6 = 3.01$

$\beta_1 = 2.145 \times 10^{-4}$ $\beta_2 = 1.4235 \times 10^{-3}$ $\beta_3 = 1.274 \quad \times \ 10^{-3}$

$\beta_4 = 2.5675 \times 10^{-3}$ $\beta_5 = 7.475 \quad \times \ 10^{-4}$ $\beta_6 = 2.73 \quad \times \ 10^{-4}$

$\alpha_f = -2.6 \quad \times \ 10^{-5}/°C$ $\alpha_m = -2.3 \quad \times \ 10^{-4}/°C$ $Q_f = 0.925$

$Q_m = 0.075$ $c_f = 0.3$ MW-sec/$°C$ $c_m = 12$ MW-sec/$°C$

$h_r = 2.482 \times 10^{-14}$ MW/$°C^4$ $H = 1.5 \times 10^{-14}$ MW.

Figures 3.10A through 3.10D show that the two forms of the reactor kinetics equations give essentially the same results for this case. The small differences between the results can be attributed to the fact that $\delta k(t)$ and ℓ were taken to be equal, respectively, to $\rho(t)$ and Λ.* Computing times for this excursion were 0.75 minute using Eqs. 3.42 and 1.08 minutes using Eq. 3.43. The difference in computing times is due to the fact that the higher derivatives of $C_i(t)$ using Eqs. 3.43 contain more terms than those using Eq. 3.42.

Since the results in this section and in Section 3.2 indicate that there are no significant differences between solutions obtained using the equations based on ℓ and those based on Λ, and since the equations based on Λ are mathematically more simple than those based on ℓ, it is recommended that the equations based on Λ be used in all reactor kinetic analyses.

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* $\delta k(t)$ and ℓ differ from $\rho(t)$ and Λ by a factor of $k(t)$ which is always close to unity.
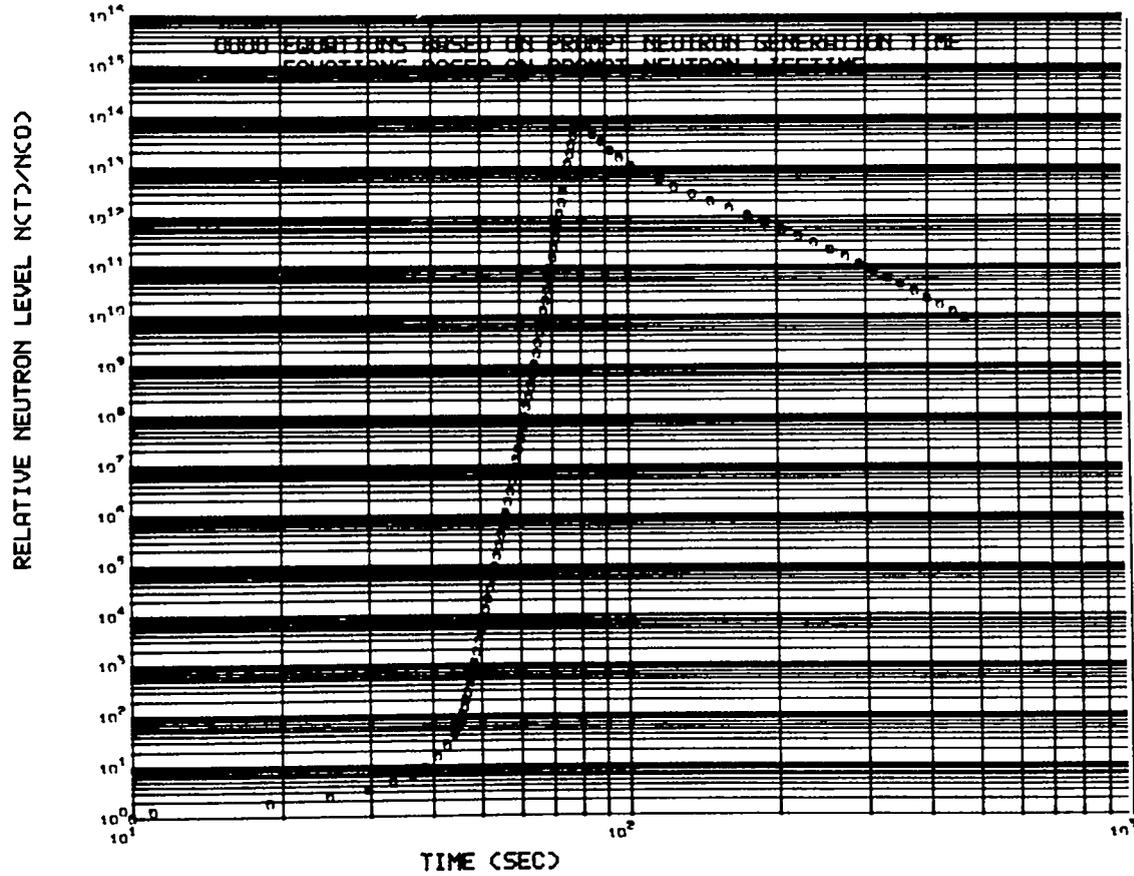
FIG 3.10A   COMPARISON OF RESULTS USING THE TWO FORMS
            OF THE REACTOR KINETICS EQUATIONS.

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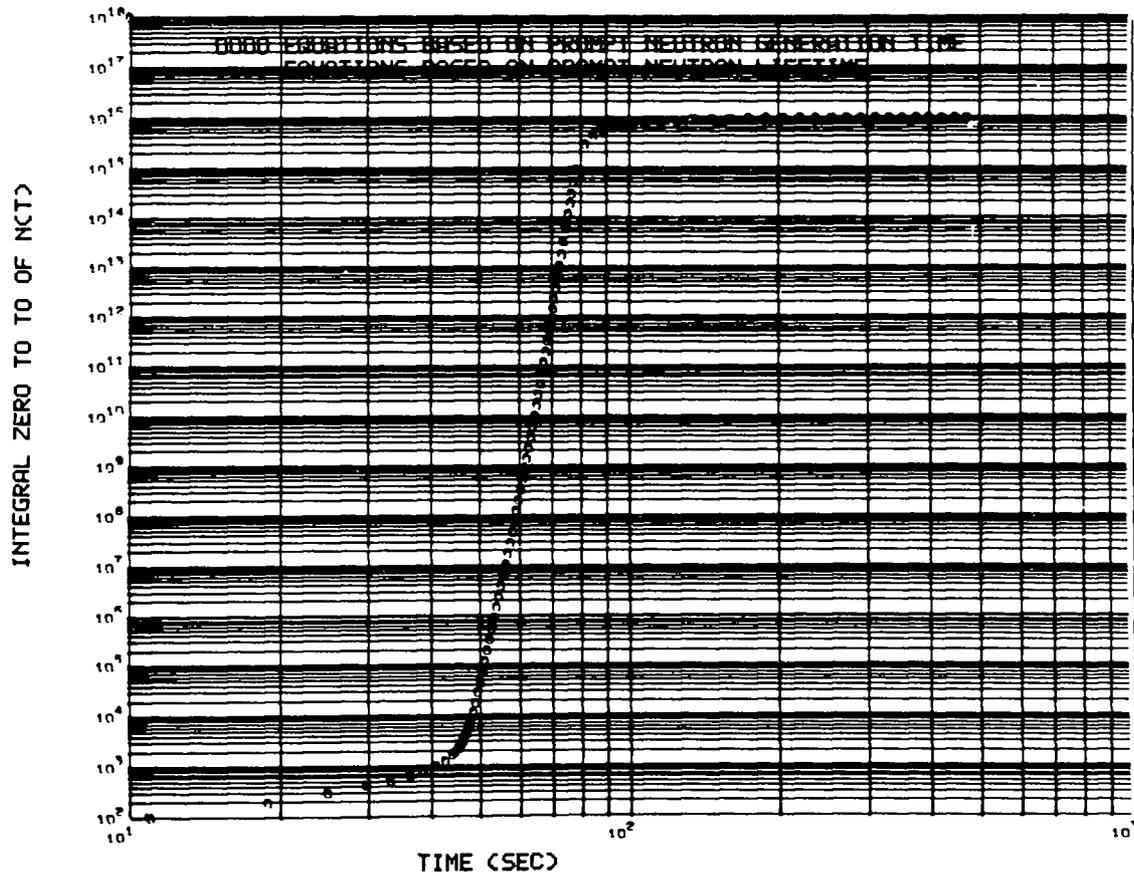
FIG 3.10B COMPARISON OF RESULTS USING THE TWO FORMS
           OF THE REACTOR KINETICS EQUATIONS.
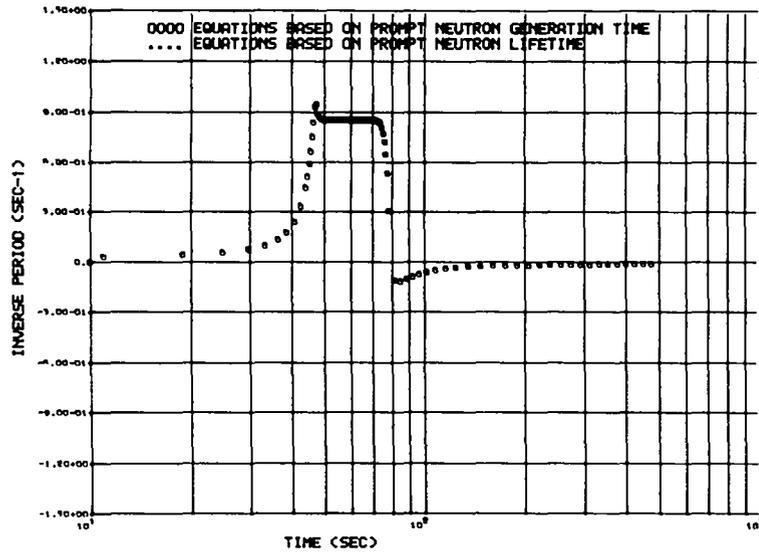
FIG 3.10C COMPARISON OF RESULTS USING THE TWO FORMS
OF THE REACTOR KINETICS EQUATIONS.
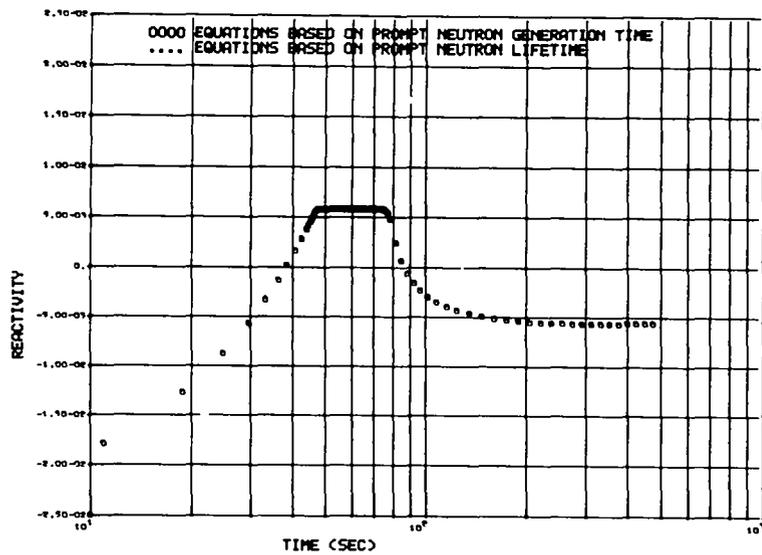
FIG 3.10D COMPARISON OF RESULTS USING THE TWO FORMS
OF THE REACTOR KINETICS EQUATIONS.

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4. SUMMARY AND CONCLUSIONS

A method for obtaining approximate solutions to the nonlinear reactor kinetics equations was described, and its properties were investigated. This method, known as continuous analytic continuation, has several advantages over numerical methods currently in use. The most important of these advantages is that the method yields an analytic criterion for the magnitude of the time step, and the criterion is such that the time step automatically expands or contracts, depending on the behavior of the neutron level in the interval. The use of this criterion to determine the time step guarantees that the error in the results increases at most linearly with the number of time steps.

Approximate solutions by continuous analytic continuation were compared with analytic solutions to the reactor kinetics equations for some of the few special cases in which analytic solutions are known. The agreement between the approximate and analytic solutions was excellent and the error accumulation in the approximate method was found in all cases to be within the limits predicted by the theory. Comparisons were then made with a numerical integration method for several cases in which analytic solutions are not available. The agreement between the two methods was good, and it was found that continuous analytic continuation was faster than the numerical integration method. Some properties predicted for continuous analytic continuation were also verified in these calculations. Comparisons also were made with experimental transients in the Godiva and SPERT I reactors. The calculated results using continuous analytic continuation were in good agreement with experiment in the region where the feedback model used was valid.

Solutions using the two forms of the reactor kinetics equations were found to be identical for all practical purposes. Since the form based on the prompt neutron generation time is simpler from a mathematical viewpoint, this form is to be preferred over the form based on the prompt neutron lifetime.

5. RECOMMENDATIONS FOR FUTURE WORK

The proposed method was used successfully to describe open-loop reactor dynamics, that is those problems in which there is no coolant flow or those problems in which the excursions are fast enough so that the dynamics of the coolant loop need not be considered. The possibility of using the method to describe closed-loop dynamics should be investigated. This will involve incorporating additional differential equations into the feedback equations and describing transport delays.

In those cases with thermal feedback considered in Section 3, temperature coefficients of reactivity and heat capacities were taken to be constant. For systems in which these quantities are sensitive to changes in temperature, the constant approximation will lead to erroneous results. Temperature coefficients of reactivity and heat capacities which are specified functions of temperature can be handled easily by the method. This capability will be incorporated into the computer program in the near future.

The possibility of extending the method to functions of two variables (time and one space dimension) should be investigated. This capability is needed for solving the coupled neutronics-hydrodynamics equations which are required to describe effects of inertia.

APPENDIX A

DERIVATION OF SPACE-, ENERGY-, AND DIRECTION-AVERAGED
REACTOR KINETICS EQUATIONS

In this appendix, the space-, energy-, and direction-averaged reactor kinetics equations are derived from the time-dependent Boltzmann equation for reactor systems in which fuel is stationary.

Let $n(\bar{r}, \bar{\Omega}, E, t)$ denote the number of neutrons per unit volume, solid angle, and energy at position $\bar{r}$, moving in the direction of the unit vector $\bar{\Omega}$ with energy E at time t. The number of neutrons $n(\bar{r}, \bar{\Omega}, E, t) \Delta\bar{r} \, \Delta\omega \, \Delta E$ in the cell $\Delta\bar{r} \, \Delta\omega \, \Delta E$ ($\Delta\bar{r}$ is a volume element and $\Delta\omega$ is an element of solid angle about the unit vector $\bar{\Omega}$) centered at $\bar{r}, \bar{\Omega}, E$ changes with time due to the following processes:

- (a) collisions in the cell $\Delta\bar{r} \, \Delta\omega \, \Delta E$ which result in absorption or scattering out of the cell,

- (b) collisions in a cell $\Delta\bar{r} \, \Delta\omega' \, \Delta E'$ which result in neutrons being scattered into $\Delta\bar{r} \, \Delta\omega \, \Delta E$,

- (c) fissions in a cell $\Delta\bar{r} \, \Delta\omega' \, \Delta E'$ which result in prompt neutrons being produced in the cell $\Delta\bar{r} \, \Delta\omega \, \Delta E$,

- (d) net streaming or leakage of neutrons out of $\Delta\bar{r}$ due to their motion,

- (e) neutrons produced in $\Delta\bar{r} \, \Delta\omega \, \Delta E$ due to radioactive decay of delayed neutron precursors, and

- (f) artificial or extraneous neutron sources producing neutrons in $\Delta\bar{r} \, \Delta\omega \, \Delta E$.

Let $C_i(\bar{r}, t)$ denote the number of delayed neutron precursors of type i per unit volume at position $\bar{r}$ and at time t. Then the number of delayed neutron precursors $C_i(\bar{r}, t) \, \Delta\bar{r}$ in the volume element $\Delta\bar{r}$ centered at $\bar{r}$ changes with time due to the following processes:

(g)   production in $\Delta \overline{r}$ due to fission, and

(h). losses in $\Delta \overline{r}$ due to radioactive decay.

The mathematical expressions which describe processes (a) through (h) can be shown to be those given below.

(a) $\sigma_t(\overline{r},E,t) \; \psi(\overline{r},\overline{\Omega},E,t) \; \Delta \overline{r} \; \Delta \omega \; \Delta E$

(b) $\int d\omega' \; dE' \; \sigma_s(\overline{r},\overline{\Omega}' \rightarrow \overline{\Omega}, \; E' \rightarrow E,t) \; \psi(\overline{r},\overline{\Omega}',E',t) \; \Delta \overline{r} \; \Delta \omega \; \Delta E$

(c) $\int d\omega' \; dE' \; \chi^p(E',E)(1 - \beta) \; \nu(E') \; \sigma_f(\overline{r},E',t) \; \psi(\overline{r},\overline{\Omega}',E',t) \; \Delta \overline{r} \; \Delta \omega \; \Delta E$

(d) $\overline{\Omega} \cdot \nabla \; \psi(\overline{r},\overline{\Omega},E,t) \; \Delta \overline{r} \; \Delta \omega \; \Delta E$

(e) $\sum_i \lambda_i \; C_i(\overline{r},t) \; \chi^d_i(E) \; \Delta \overline{r} \; \Delta \omega \; \Delta E$

(f) $S(\overline{r},\overline{\Omega},E,t) \; \Delta \overline{r} \; \Delta \omega \; \Delta E$

(g) $\int d\omega' \; dE' \; \beta_i \; \nu(E') \; \sigma_f(\overline{r},E',t) \; \psi(\overline{r},\overline{\Omega}',E',t) \; \Delta \overline{r}$

(h) $\lambda_i \; C_i(\overline{r},t) \; \Delta \overline{r}$

where

$\psi(\overline{r},\overline{\Omega},E,t) \equiv v(E) \; n(\overline{r},\overline{\Omega},E,t)$ = directional neutron flux per unit volume, energy, and solid angle

$v(E)$ = velocity associated with neutrons of energy E

$\sigma_t(\overline{r},E,t)$ = total macroscopic interaction cross section (assumed to be independent of $\overline{\Omega}$)

$\sigma_s(\overline{r},\overline{\Omega}' \rightarrow \overline{\Omega},E' \rightarrow E,t)$ = macroscopic scattering cross section for scattering from $E',\overline{\Omega}'$ to $E,\overline{\Omega}$.

$\sigma_f(\overline{r},E,t)$ = macroscopic fission cross section (assumed to be independent of $\overline{\Omega}$)

$\chi^P(E',E)$ = energy spectrum of prompt fission neutrons when the neutron causing fission has energy E' (assumed to be independent of $\bar{\Omega}$), normalized so that

$$\int_0^\infty \chi^P(E',E)\, dE = 1 \qquad\qquad \text{for all } E'$$

$\chi_i^d(E)$ = energy spectrum of delayed neutrons from group i of delayed neutron precursors (assumed to independent of $\bar{\Omega}$), normalized so that

$$\int_0^\infty \chi_i^d(E)\, dE = 1 \qquad\qquad \text{for all } i$$

$S(\bar{r},\bar{\Omega},E,t)$ = directional extraneous source density

$\nu(E)$ = average number of neutrons emitted per fission when the neutron causing fission has energy E,

λ_i = decay constant of group i of delayed neutron precursors,

β_i = fraction of fission neutrons emitted by group i of delayed neutron precursors, and

β = total delayed neutron fraction, that is

$$\beta = \sum_i \beta_i.$$

By equating the time rate of change of $n(\bar{r},\bar{\Omega},E,t)\,\Delta\bar{r}\,\Delta\omega\,\Delta E$ and $C_i(\bar{r},t)\Delta\bar{r}$ to the rate of production minus the removal rate, the following equations are obtained.

$$\frac{1}{v(E)}\frac{\partial\psi}{\partial t}(\bar{r},\bar{\Omega},E,t) = -\bar{\Omega}\cdot\nabla\psi(\bar{r},\bar{\Omega},E,t) - \sigma_t(\bar{r},E,t)\,\psi(\bar{r},\bar{\Omega},E,t)$$

$$+ \int d\omega'\, dE'\left[\sigma_s(\bar{r},\bar{\Omega}'\to\bar{\Omega},E'\to E,t) + (1-\beta)\,\nu(E')\,\chi^P(E',E)\,\sigma_f(\bar{r},E',t)\right]\psi(\bar{r},\bar{\Omega}',E',t)$$

$$+ S(\bar{r},\bar{\Omega},E,t) + \sum_i \lambda_i\,C_i(\bar{r},t)\,\chi_i^d(E) \qquad\qquad (A.1)$$

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$$\frac{\partial C_i}{\partial t}(\bar{r},t) = \int d\omega' \, dE' \, \beta_i \, \nu(E') \, \sigma_f(\bar{r},E',t) \, \psi(\bar{r},\bar{\Omega}',E',t) - \lambda_i \, C_i(\bar{r},t).$$

(A.2)

Equations A.1 and A.2 apply to a reactor system in which fuel is stationary, since transport of delayed neutron precursors was not considered. It was assumed that both prompt and delayed neutron sources from fission, as well as σ_t and σ_f are isotropic. In addition, it was assumed that the decay of a delayed neutron precursor results in the production of exactly one neutron and also that scattering is instantaneous.

Equations A.1 and A.2 describe the behavior of the noncritical system. Consider now the delayed critical case, that is, the steady state case without an extraneous source. For this case, the above equations become

$$-\bar{\Omega} \cdot \nabla \phi_o(\bar{r},\bar{\Omega},E) - \sigma_{t,o}(\bar{r},E) \, \phi_o(\bar{r},\bar{\Omega},E)$$

$$+ \int d\omega' \, dE' \left[\sigma_{s,o}(\bar{r},\bar{\Omega}' \to \bar{\Omega}, E' \to E) + (1-\beta) \, \nu(E') \, \chi^p(E',E) \, \sigma_{f,o}(\bar{r},E') \right] \phi_o(\bar{r},\bar{\Omega}',E')$$

$$+ \sum_i \lambda_i \, C_{i,o}(\bar{r}) \, \chi_i^d(E) = 0 \qquad (A.3)$$

$$\int d\omega' \, dE' \, \beta_i \, \nu(E') \, \sigma_{f,o}(\bar{r},E') \, \phi_o(\bar{r},\bar{\Omega}',E') - \lambda_i \, C_{i,o}(\bar{r}) = 0 \qquad (A.4)$$

where ϕ_o and $C_{i,o}$ are the directional flux and delayed neutron precursor distributions, respectively, at delayed critical. The values of the various cross sections at delayed critical are denoted by the subscript o.

Equations A.3 and A.4 can be combined to yield

$$-\bar{\Omega} \cdot \nabla \, \phi_o(\bar{r},\bar{\Omega},E) - \sigma_{t,o}(\bar{r},E) \, \phi_o(\bar{r},\bar{\Omega},E) + \int d\omega' \, dE' \Big\{ \sigma_{s,o}(\bar{r},\bar{\Omega}' \to \bar{\Omega}, E' \to E)$$

$$+ \left[\chi^p(E',E)(1-\beta) + \sum_i \chi_i^d(E) \, \beta_i \right] \nu(E') \, \sigma_{f,o}(\bar{r},E') \Big\} \, \phi_o(\bar{r},\bar{\Omega}',E')$$

$$= 0 \qquad (A.5)$$

For reasons which will become clear later in the development, the adjoint of Eq. A.5 is more useful for the purpose at hand. The adjoint equation is obtained formally by replacing $-\overline{\Omega}$ by $\overline{\Omega}$ in the first term and transposing the kernel in the integral term:

$$\overline{\Omega} \cdot \nabla \phi_o^*(\overline{r},\overline{\Omega},E) - \sigma_{t,o}(\overline{r},E) \phi_o^*(\overline{r},\overline{\Omega},E) + \int d\omega' \, dE' \left\{ \sigma_{s,o}(\overline{r},\overline{\Omega}{\rightarrow}\overline{\Omega}',E{\rightarrow}E') \right.$$

$$+ \left[\chi^p (E,E')(1 - \beta) + \sum_i \chi_i^d (E')\beta_i \right] \nu(E) \left. \sigma_{f,o}(\overline{r},E) \right\} \phi_o^*(\overline{r},\overline{\Omega}',E')$$

$$= 0 \tag{A.6}$$

In Eq. A.6, ϕ_o^* is the neutron importance function at delayed critical. The concept of neutron importance is discussed in Reference 25.

In order to simplify the notation in the manipulations which follow, the following substitutions are made.

$$S(\overline{r},\overline{\Omega},E,t) = S(\overline{x},t)$$
$$\psi(\overline{r},\overline{\Omega},E,t) = \psi(\overline{x},t)$$
$$\phi_o(\overline{r},\overline{\Omega},E) = \phi_o(\overline{x})$$
$$\sigma(\overline{r},\overline{\Omega},E,t) \equiv \sigma(\overline{r},E,t) = \sigma(\overline{x},t)$$
$$\sigma(\overline{r},\overline{\Omega}'{\rightarrow}\overline{\Omega},E'{\rightarrow}E,t) = \sigma(\overline{r},y'{\rightarrow}y,t)$$

$$dy = d\omega \, dE \qquad d\overline{x} = d\overline{r} \, d\omega \, dE = d\overline{r} \, dy \tag{A.7}$$

With the above substitutions, Eqs. A.1 and A.6 become, respectively,

$$\frac{1}{v(E)} \frac{\partial \psi}{\partial t} (\overline{x},t) = -\overline{\Omega} \cdot \nabla \psi(\overline{x},t) - \sigma_t(\overline{x},t) \psi(\overline{x},t)$$

$$+ \int dy^{\prime} \left[\sigma_s(\overline{r},y^{\prime} \to y,t) + (1 - \beta) v(E^{\prime}) \chi^p (E^{\prime},E) \sigma_f(\overline{x}^{\prime},t) \right] \psi(\overline{x}^{\prime},t)$$

$$+ S(\overline{x},t) + \sum_i \lambda_i C_i(\overline{r},t) \chi_i^d (E) \qquad \qquad (A.8)$$

$$\overline{\Omega} \cdot \nabla \phi_o^*(\overline{x}) - \sigma_{t,o}(\overline{x}) \phi_o^*(\overline{x}) + \int dy^{\prime} \Big\{ \sigma_{s,o}(\overline{r},y \to y^{\prime})$$

$$+ \left[\chi^p (E,E^{\prime})(1 - \beta) + \sum_i \chi_i^d (E^{\prime})\beta_i \right] v(E) \sigma_{f,o}(\overline{x}) \Big\} \phi_o^*(\overline{x}^{\prime}) = 0 \qquad (A.9)$$

Multiplying A.8 by $\phi_o^*(\overline{x})$ and A.9 by $\psi(\overline{x},t)$, subtracting the resulting equations, and combining a few terms gives

$$\frac{\partial}{\partial t} \left[\frac{\phi_o^*(\overline{x}) \psi(\overline{x},t)}{v(E)} \right] = -\nabla \cdot \left[\overline{\Omega} \phi_o^*(\overline{x}) \psi(\overline{x},t) \right] - \delta \sigma_t(\overline{x},t) \phi_o^*(\overline{x}) \psi(\overline{x},t)$$

$$+ \int dy^{\prime} \left[\sigma_s(\overline{r},y^{\prime} \to y,t) + (1 - \beta) v(E^{\prime}) \chi^p (E^{\prime},E) \sigma_f(\overline{x}^{\prime},t) \right] \phi_o^*(\overline{x}) \psi(\overline{x}^{\prime},t)$$

$$- \int dy^{\prime} \Big\{ \sigma_{s,o}(\overline{r},y \to y^{\prime}) + \left[\chi^p (E,E^{\prime})(1 - \beta) + \sum_i \chi_i^d (F^{\prime})\beta_i \right] v(E) \sigma_{f,o}(\overline{x}) \Big\} \cdot$$

$$\phi_o^*(\overline{x}^{\prime}) \psi(\overline{x},t) + S(\overline{x},t) \phi_o^*(\overline{x}) + \sum_i \lambda_i C_i(\overline{r},t) \chi_i^d (E) \phi_o^*(\overline{x})$$

$$\qquad \qquad (A.10)$$

where

$$\delta \sigma_t (\overline{x},t) = \sigma_t(\overline{x},t) - \sigma_{t,o}(\overline{x}).$$

If the time dependence of $\psi(\overline{x},t)$ is separable, that is,

$$\psi(\overline{x},t) = N(t) \; \phi(\overline{x}) \tag{A.11}$$

then, on integrating over all space, energy, and solid angle, Eq. A.10 becomes

$$\frac{dN(t)}{dt} \int d\overline{x} \; \frac{\phi_o^*(\overline{x}) \; \phi(\overline{x})}{v(\omega)} = -N(t) \int dy \int dr \; \nabla \cdot [\overline{\Omega} \; \phi_o^*(\overline{x}) \; \phi(\overline{x})]$$

$$-N(t) \int d\overline{x} \; \delta\sigma_t(\overline{x},t) \; \phi_o^*(\overline{x}) \; \phi(\overline{x}) + N(t) \int d\overline{r} \; dy \; dy' \; \cdot$$

$$\left[\sigma_s(\overline{r},y'{\to}y,t) + (1 - \beta) \; \nu(E') \; \chi^P(E',E) \; \sigma_f(\overline{x}',t) \right] \phi_o^*(\overline{x}) \; \phi(\overline{x}')$$

$$- N(t) \int d\overline{r} \; dy \; dy' \left\{ \sigma_{s,o}(\overline{r},y{\to}y') + \left[(1 - \beta) \; \chi^P(E,E') + \sum_i \beta_i \; \chi_i^d(E') \right] \nu(E) \; \sigma_{f,o}(\overline{x}) \right\}$$

$$\cdot \phi_o^*(\overline{x}') \; \phi(\overline{x}) + \int d\overline{x} \; S(\overline{x},t) \; \phi_o^*(\overline{x}) + \sum_i \lambda_i \int d\overline{x} \; C_i(\overline{r},t) \chi_i^d(E) \; \phi_o^*(\overline{x}).$$

$$\tag{A.12}$$

By use of the divergence theorem, the integral over space in the first term on the right side of Eq. A.12 can be written

$$\int_{volume} d\overline{r} \; \nabla \cdot [\overline{\Omega} \; \phi_o^*(\overline{x}) \; \phi(\overline{x})] = \int_{surface} (\overline{n} \cdot \overline{\Omega}) \; \phi_o^* \; \phi \; dA \tag{A.13}$$

where $\overline{n}$ is the outward normal to the surface of a large sphere enclosing the system. If there are no neutrons entering from outside the sphere enclosing the system, that is,

$\phi = 0$ for $-1 \leq \overline{n} \cdot \overline{\Omega} \leq 0$, then by necessity the importance of neutrons leaving the sphere enclosing the system is zero, that is, $\phi^* = 0$ for $0 \leq \overline{n} \cdot \overline{\Omega} \leq 1$. Thus, the integral over the surface of the sphere of the normal component of the product $\phi^* \phi$ vanishes and Eq. A.13 is identically equal to zero. This is one of the reasons for using ϕ^* as a weighting function.

Since E, E', ω, and ω' are dummy variables in the integration, it can be seen that

$$\int d\overline{r} \, dy \, dy' \left\{ \sigma_{s,o}(\overline{r}, y \to y') + \left[(1 - \beta) \chi^p(E, E') + \sum_i \beta_i \chi^d_i(F') \right] \nu(E) \, \sigma_{f,o}(\overline{x}) \right\}$$

$$\cdot \, \phi_o^*(\overline{x'}) \, \phi(\overline{x}) \equiv \int d\overline{r} \, dy' \, dy \left\{ \sigma_{s,o}(\overline{r}, y' \to y) + \right.$$

$$\left. \left[(1 - \beta) \chi^p(E', E) + \sum_i \beta_i \chi^d_i(E) \right] \nu(E') \, \sigma_{f,o}(\overline{x'}) \right\} \, \phi_o^*(\overline{x}) \, \phi(\overline{x'}).$$

$$(A.14)$$

With Eqs. A.13 and A.14, Eq. A.12 becomes

$$\frac{dN(t)}{dt} \int d\overline{x} \, \frac{\phi_o^*(\overline{x}) \, \phi(\overline{x})}{\nu(E)} = N(t) \int d\overline{r} \, dy \, dy' \left\{ \delta\sigma_s(\overline{r}, y' \to y, t) + \left[(1 - \beta) \chi^p(E', E) \right. \right.$$

$$\left. + \sum_i \beta_i \chi^d_i(E) \right] \nu(E') \, \delta\sigma_f(\overline{x'}, t) \right\} \, \phi_o^*(\overline{x}) \, \phi(\overline{x'})$$

$$- N(t) \int d\overline{x} \, \delta\sigma_t(\overline{x}, t) \, \phi_o^*(\overline{x}) \, \phi(\overline{x}) - N(t) \int d\overline{r} \, dy \, dy' \sum_i \beta_i \chi^d_i(E).$$

$$\nu(E') \, \sigma_f(\overline{x'}, t) \, \phi_o^*(\overline{x}) \, \phi(\overline{x'}) + \int d\overline{x} \, S(\overline{x}, t) \, \phi_o^*(\overline{x})$$

$$+ \sum_i \lambda_i \int d\overline{x} \, C_i(\overline{r}, t) \, \chi^d_i(E) \, \phi_o^*(\overline{x})$$

$$(A.15)$$

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where

$$\delta\sigma_s(\bar{r}, y' \to y, t) = \sigma_s(\bar{r}, y' \to y, t) - \sigma_{s,o}(\bar{r}, y' \to y)$$

$$\delta\sigma_f(\bar{x}, t) = \sigma_f(\bar{x}, t) - \sigma_{f,o}(\bar{x}).$$

The corresponding equations for the delayed neutron precursors are obtained by multiplying Eq. A.2 by $\phi_o^*(\bar{x}) \chi_i^d(E)$ and integrating over all space, energy, and solid angle to obtain

$$\frac{\partial}{\partial t} \int d\bar{x}\, C_i(\bar{r}, t)\, \chi_i^d(E)\, \phi_o^*(\bar{x}) = N(t)\, \beta_i \int d\bar{r}\, dy\, dy'\, \chi_i^d(E) \cdot$$

$$\nu(E')\, \sigma_f(\bar{x}', t)\, \phi_o^*(\bar{x})\, \phi(\bar{x}') - \lambda_i \int d\bar{x}\, C_i(\bar{r}, t)\, \chi_i^d(E)\, \phi_o^*(\bar{x}). \qquad (A.16)$$

In order to obtain the reduced equations, the following definitions are made:

$$F(t) = \int d\bar{r}\, dy\, dy'\, \left[(1 - \beta)\, \chi^p(E', E) + \sum_i \beta_i\, \chi_i^d(E) \right] \nu(E')\, \sigma_f(\bar{x}', t)$$

$$\cdot\ \phi_o^*(\bar{x})\, \phi(\bar{x}') \qquad (A.17)$$

The function F(t) can be regarded as the fission source integrated over space, energy, and direction using ϕ_o^* as a weighting function. Note that the prompt and delayed fission neutrons are weighted by their appropriate energy spectra. F(t) is used in some of the definitions given below.

$$\rho(t) = \frac{1}{F(t)} \left\{ \int d\bar{r}\, dy\, dy'\, \left(\delta\sigma_s(r, y' \to y, t) + \left[(1 - \beta)\, \chi^p(E', E) + \sum_i \beta_i\, \chi_i^d(E) \right] \cdot \right. \right.$$

$$\left. \nu(E')\, \delta\sigma_f(\bar{x}', t) \right)\, \phi_o^*(\bar{x})\, \phi(\bar{x}') - \int d\bar{x}\, \delta\sigma_t(\bar{x}, t)\, \phi_o^*(\bar{x})\, \phi(\bar{x}) \left. \right\} \qquad (A.18)$$

Equation A.18 expresses the reactivity $\rho(t)$ in terms of changes (relative to the initial steady state case) in the various interaction rates integrated over space, energy, and direction using ϕ_o^* as a weighting function. F(t) occurs as a normalization factor in this definition.

$$\Lambda(t) = \frac{1}{F(t)} \int d\overline{x} \, \frac{\phi_o^*(\overline{x}) \, \phi(\overline{x})}{v(E)} \qquad\qquad (A.19)$$

Equation A.19 is the definition of the prompt neutron generation time Λ. Its significance can be seen by considering the monoenergetic, isotropic case (no E or $\overline{\Omega}$ dependence). For this case, Eq. A.19 becomes $\Lambda(t) = 1/v\nu \, \overline{\sigma}_f(t)$ where

$$\overline{\sigma}_f(t) = \int d\overline{r} \, \sigma_f(\overline{r},t) \, \phi_o^*(\overline{r}) \, \phi(\overline{r}) \Big/ \int d\overline{r} \, \phi_o^*(\overline{r}) \, \phi(\overline{r}).$$

The quantity $1/v \, \overline{\sigma}_f$ is the average time between collisions which result in fission. Since ν neutrons are produced per fission, $\Lambda(t) = 1/\nu v \overline{\sigma}_f(t)$ can be interpreted as 1/production rate. The prompt neutron generation time is time dependent because σ_f may vary with time during an excursion in which the temperature changes. The variation of Λ with temperature is so small, however, that it is neglected. In a reactor controlled by fuel rods, σ_f and thus Λ, will vary with time if a control rod is inserted or withdrawn during the transient. On the other hand, in a reactor controlled by poison rods, Λ is essentially independent of control rod position. For a more detailed discussion of this point, see Reference 26. In practice, the time dependence of Λ is always neglected.

$$\overline{\beta}_i(t) = \frac{\beta_i}{F(t)} \int d\overline{r} \, dy \, dy' \, \chi_i^d(E) \, \nu(E') \, \sigma_f(\overline{x}',t) \, \phi_o^*(\overline{x}) \, \phi(\overline{x}') \qquad\qquad (A.20)$$

Equation A.20 defines the effective delayed neutron fraction $\overline{\beta}_i$ for group i of delayed neutron precursors. $\overline{\beta}_i$ is equal to the actual fraction β_i, multiplied by a factor which takes into account the fact that delayed fission neutrons have a different energy spectrum than prompt fission neutrons. Since σ_f occurs in both numerator and denominator of the factor which multiplies β_i, $\overline{\beta}_i$ is not sensitive to changes in σ_f with time. Thus, in practice $\overline{\beta}_i$ is considered to be independent of time.

$$\bar{\beta}(t) = \sum_i \bar{\beta}_i(t) = \text{total effective delayed neutron fraction} \qquad (A.21)$$

$$\bar{C}_i(t) = \int d\bar{x}\, C_i(\bar{r},t)\, \chi_i^d(E)\, \phi_o^*(\bar{x}) \Big/ \int d\bar{x}\, \frac{\phi_o^*(\bar{x})\, \phi(\bar{x})}{v(E)} \qquad (A.22)$$

The effective delayed neutron precursor level $\bar{C}_i(t)$ for group i of delayed neutron precursors is defined as the integral over space, energy, and direction of $C_i(\bar{r},t)$ using χ_i^d and ϕ_o^* as weighting functions. The integral in the denominator in Eq. A.22 is a normalization factor. Since $\phi(\bar{x}) = v(E)\, n(\bar{x})$, this normalization factor is the integral of $n(\bar{x})$ over space, energy, and direction using ϕ_o^* as a weighting function:

$$\bar{S}(t) = \int d\bar{x}\, S(\bar{x},t)\, \phi_o^*(\bar{x}) \Big/ \int d\bar{x}\, \frac{\phi_o^*(\bar{x})\, \phi(\bar{x})}{v(E)} \qquad (A.23)$$

The effective extraneous source $\bar{S}(t)$ is defined as the integral over space, energy, and direction of $S(\bar{x},t)$ using ϕ_o^* as a weighting function. The integral in the denominator in Eq. A.23 is again a normalization factor as described above for $\bar{C}_i(t)$.

Substitution of Eqs. A.18 through A.23 into Eqs. A.15 and A.16 gives

$$\frac{dN(t)}{dt} = \frac{\rho(t) - \bar{\beta}}{\Lambda}\, N(t) + \sum_i \lambda_i\, \bar{C}_i(t) + \bar{S}(t) \qquad (A.24)$$

$$\frac{d\bar{C}_i(t)}{dt} = \frac{\bar{\beta}_i}{\Lambda}\, N(t) - \lambda_i\, \bar{C}_i(t) \qquad i=1, \ldots, I \qquad (A.25)$$

The time dependence of Λ, $\bar{\beta}_i$, and $\bar{\beta}$ in Eqs. A.24 and A.25 has been neglected for reasons discussed above. Since the assumption of constant Λ is more justifiable for reactors which are controlled by poison rods, Eqs. A.24 and A.25 are more appropriate for these reactors.

An alternate form of the reactor kinetics equations can be obtained by making use of the multiplication factor k(t) and the prompt neutron lifetime ℓ. By definition, the relation between k(t) and ρ(t) is

$$[k(t) - 1]/k(t) = \rho(t). \tag{A.26}$$

An expression for k(t) in terms of changes in the various cross sections can be obtained by combining Eqs. A.26 and A.18. The prompt neutron lifetime is basically given by $1/v\bar{\sigma}_a$ where $\bar{\sigma}_a$ is the effective macroscopic absorption cross section (including leakage). Thus, ℓ can be interpreted as 1/destruction rate. Since k(t) is equal to production rate/destruction rate, it is seen that

$$k(t) = \ell(t)/\Lambda(t). \tag{A.27}$$

Substitution of Eqs. A.26 and A.27 into Eqs. A.24 and A.25 yields the alternate form of the reactor kinetics equations

$$\frac{dN(t)}{dt} = \left[\frac{(1 - \bar{\beta}) \; k(t) - 1}{\ell} \right] N(t) + \sum_1 \lambda_i \; \bar{C}_i(t) + \bar{S}(t) \tag{A.28}$$

$$\frac{d\bar{C}_i(t)}{dt} = \left[\frac{\bar{\beta}_i \; k(t)}{\ell} \right] N(t) - \lambda_i \; \bar{C}_i(t) \qquad i=1, \ldots, I \tag{A.29}$$

Again the usual practice is to neglect the time dependence of ℓ in Eqs. A.28 and A.29. In this case, the assumption of constant ℓ is more justifiable for reactors employing fueled control rods.

Since reactors employing poison for control are more common than those using fueled control rods, the equations based on Λ are used in this study. The equations which describe N(t) are nonlinear in both cases because of the term ρ(t) N(t) in Eq. A.24 and the term k(t) N(t) in Eq. A.28. These

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terms are nonlinear because in problems involving feedback, $\rho(t)$ and $k(t)$ are functions of $N(t)$. The delayed neutron precursor equations using a constant prompt neutron generation time Λ (Eq. A.25) are linear, while those using a constant prompt neutron lifetime ℓ (Eq. A.29) are nonlinear because of the $k(t) N(t)$ term. Thus, from a mathematical viewpoint, the system of equations based on Λ (Eqs. A.24 and A.25) is simpler than the system of equations based on ℓ (Eqs. A.28 and A.29). The method of continuous analytic continuation can be used on both systems of equations, however.

In the above derivation, the conventional kinetics equations for reactors in which fuel is stationary were obtained in the time separable case from the time dependent transport equation. Since the derivation is exact for the time separable case, precise definitions of the various quantities appearing in the reduced equations were obtained. Lewins[27] has obtained the conventional **form** of the kinetics equations for the general case of mobile fuel without the assumption of separability. In this case, all quantities appearing in the reduced equations are weighted averages over all variables except time. One of the weighting functions in the nonseparable case is $\psi(\bar{r},\bar{\Omega},E,t)$ and hence, this function must be known before some weighted averages (such as Λ) used in the reduced equations can be calculated. However, it is much more difficult to determine $\psi(\bar{r},\bar{\Omega},E,t)$ than it is to solve the reduced kinetic equations. Furthermore, if the behavior of $\psi(\bar{r},\bar{\Omega},E,t)$ were known, there would be no need for reduced equations, since $\psi(\bar{r},\bar{\Omega},E,t)$ already contains all the necessary information.

APPENDIX B

EXPRESSIONS FOR HIGHER ORDER DERIVATIVES OF VARIABLES
APPEARING IN REACTOR KINETICS EQUATIONS

The method of continuous analytic continuation requires the computation of the derivatives, to order $K + 1$, of the variables appearing in the reactor kinetics equations for each time step. For solution on a digital computer, the establishment of general expressions for these derivatives is convenient.

Consider first the reactor kinetics equations based on the prompt neutron generation time. These equations (Eqs. 2.1-2.3, Section 2.2) are reproduced below.

$$N^{(1)}(t) = \Lambda^{-1}[\rho(t) - \beta] N(t) + \sum_{i=1}^{I} \lambda_i C_i(t) + S(t) \tag{B.1}$$

$$C_i^{(1)}(t) = \Lambda^{-1} \beta_i N(t) - \lambda_i C_i(t) \qquad i=1, \ldots, I \tag{B.2}$$

$$\rho(t) = I(t) + F(t) \tag{B.3}$$

The k+1th derivative of $N(t)$ can be established by mathematical induction by successive differentiation of Eq. B.1. The result is

$$N^{(k+1)}(t) = \Lambda^{-1}[\rho(t) - \beta] N^{(k)}(t) + \frac{\Lambda^{-1}k}{1!} \rho^{(1)} N^{(k-1)}$$

$$+ \frac{\Lambda^{-1} k(k-1)}{2!} \rho^{(2)} N^{(k-2)} + \ldots + \Lambda^{-1} \rho^{(k)} N$$

$$+ \sum_{i=1}^{I} \lambda_i C_i^{(k)}(t) + S^{(k)}(t) \tag{B.4}$$

for $0 \le k \le K$.

In terms of the binomial coefficients defined by

$$B_n^k = \frac{k(k-1) \cdots (k-n+1)}{n!} = \frac{k!}{n!(k-n)!} , \tag{B.5}$$

Eq. B.4 can be written in a more compact form:

$$N^{(k+1)}(t) = \Lambda^{-1}[\rho(t) - \beta] N^{(k)}(t) + \Lambda^{-1} \sum_{n=1}^{k} B_n^k \rho^{(n)}(t) N^{(k-n)}$$

$$+ \sum_{i=1}^{I} \lambda_i C_i^{(k)}(t) + S^{(k)}(t) . \tag{B.6}$$

If the extraneous source is a constant, then $S^{(k)}(t) = 0$ for $k \geq 1$. If $S(t)$ is an analytic function of time (for example, $\sin \omega t$, $e^{\alpha t}$, or polynomial in t), the derivatives $S^{(k)}(t)$ can easily be obtained. On the other hand, if $S(t)$ is given as a table of numbers, the derivatives $S^{(k)}(t)$ can be obtained by fitting the data with a polynomial.

Since Eq. B.2 is linear, the k+1th derivative of $C_i(t)$ can be established by inspection to be

$$C_i^{(k+1)}(t) = \Lambda^{-1} \beta_i N^{(k)}(t) - \lambda_i C_i^{(k)}(t). \tag{B.7}$$

From Eq. B.3, it can be seen that the k+1th derivative of $\rho(t)$ is given formally by

$$\rho^{(k+1)}(t) = I^{(k+1)}(t) + F^{(k+1)}(t). \tag{B.8}$$

To proceed further, particular forms for the functions $I(t)$ and $F(t)$ must be assumed. The forms considered below have been included as options in the digital computer program. Since reactivity is introduced by the use of subroutines (see Appendix D), forms other than those considered below for $I(t)$ and $F(t)$ can be used either by expanding the present subroutines or by writing additional ones.

The first option for $I(t)$ is

$$I(t) = a_o + \rho_o \sin \omega t \tag{B.9}$$

which represents a sinusoidal variation in reactivity with angular frequency ω and amplitude ρ_o about a_o. This type of impressed reactivity can be produced by oscillating a rod within the reactor. The derivatives of $I(t)$ for this case are given by

$$I^{(2k)}(t) = (-1)^k (\omega)^{2k} \rho_o \sin \omega t \qquad \text{(even derivatives)} \tag{B.10}$$

$$I^{(2k-1)}(t) = (-1)^{k+1} (\omega)^{2k-1} \rho_o \cos \omega t \qquad \text{(odd derivatives)}$$

for $k \geq 1$.

A second option available for $I(t)$ is

$$I(t) = a_o + r_1(t) + r_2(t) \tag{B.11}$$

where

$$
\begin{aligned}
r_1(t) &= a_1 t & \text{for } t < t_1 \\
r_1(t) &= a_1 t_1 & t \geq t_1 \\
r_2(t) &= a_2 t^2 & t < t_2 \\
r_2(t) &= a_2 t_2^2 & t \geq t_2
\end{aligned} \tag{B.12}
$$

and a_o, a_1, a_2, t_1, and t_2 are constants to be specified. Thus, Eqs. B.11 and B.12 represent either a step input of reactivity (a_o), a ramp input of reactivity due to control rod movement at constant speed ($a_1 t$), a reactivity variation due to control rod movement at constant acceleration ($a_2 t^2$), or any combination of these three effects. Note that the terms $a_1 t$ and $a_2 t^2$ are terminated at $t = t_1$ and $t = t_2$, respectively. The derivatives of $I(t)$ for this case are obtained by inspection from Eqs. B.11 and B.12:

$$I^{(k)}(t) = r_1^{(k)}(t) + r_2^{(k)}(t)$$

where

$$\left. \begin{array}{ll} r_1^{(1)}(t) = a_1 & \\ \\ r_1^{(k)}(t) = 0 \qquad k \geq 2 \end{array} \right\} \quad t < t_1$$

$$r_1^{(k)}(t) = 0 \qquad k \geq 1 \qquad t \geq t_1$$

(B.14)

$$\left. \begin{array}{ll} r_2^{(1)}(t) = 2a_2 t & \\ \\ r_2^{(2)}(t) = 2a_2 & \\ \\ r_2^{(k)}(t) = 0 \qquad k \geq 3 \end{array} \right\} \quad t < t_2$$

$$r_2^{(k)}(t) = 0 \qquad k \geq 1 \qquad t \geq t_2 .$$

A third option available for $I(t)$ is a scram at $t = t_s$ after an initial step and/or ramp input of reactivity. This case is represented by defining $r_2(t)$ in Eq. B.11 as

$$r_2(t) = 0 \qquad\qquad t < t_s$$

$$r_2(t) = a_2(t - t_s)^2 \qquad t_s \leq t \leq t_s + \delta t$$

(B.15)

$$r_2(t) = a_2(\delta t)^2 \qquad\qquad t > t_s + \delta t$$

where $a_2(\delta t)^2$ is the worth of the scram rod. The derivatives of $r_2(t)$ for this case are given by

$$r_2^{(1)}(t) = 2a_2(t - t_s)$$

$$r_2^{(2)}(t) = 2a_2$$

$$r_2^{(k)}(t) = 0 \qquad k \geq 3$$

$$\left. \right\} \quad t_s \leq t \leq t_s + \delta t$$

$$r_2^{(k)}(t) = 0 \qquad k \geq 1 \qquad t < t_s \text{ or } t > t_s + \delta t. \tag{B.16}$$

Thermal feedback is introduced by letting

$$F(t) = \sum_{j=1}^{J} \alpha_j [T_j(t) - T_j(0)] \tag{B.17}$$

where α_j and $T_j(t)$ are the temperature reactivity coefficient and temperature, respectively, of the $\underline{j}$th region of the reactor. The derivatives of $F(t)$ are obtained by inspection from Eq. B.17

$$F^{(k)}(t) = \sum_{j=1}^{J} \alpha_j T_j^{(k)}(t) \tag{B.18}$$

for $1 \leq k \leq K + 1$.

Several options for the heat balance equations, to be used in conjunction with Eq. B.17, are available. The first is

$$c_j \, dT_j(t)/dt = Q_j \, BN(t) - P_j \qquad j=1, \ldots, J \tag{B.19}$$

which represents the case of constant power removal (P_j) from the $\underline{j}$th region. The heat capacity of region j is c_j; Q_j is the fraction $\overline{\text{of}}$ power deposited in region j; and B is a conversion factor from neutron level to power level. Note that for J = 1 and the adiabatic case ($P_j = 0$), Eq. B.19 yields

$$T(t) - T(0) = (QB/c) \int_0^t N(t') \, dt'$$

so that Eq. B.17 becomes for this case

$$F(t) = (\alpha QB/c) \int_0^t N(t') \, dt' \, . \tag{B.20}$$

Equation B.20 represents the case of reactivity feedback proportional to the integrated power or total energy release. The higher derivatives of Eq. B.19 are by inspection

$$T_j^{(k+1)}(t) = (Q_j B/c_j) \, N^{(k)}(t) \qquad j=1, \ldots, J \tag{B.21}$$

for $1 \leq k \leq K$.

The second option for the heat balance equations is

$$c_j \, dT_j(t)/dt = Q_j \, BN(t) - b_j \, T_j(t) \qquad j=1, \ldots, J \tag{B.22}$$

where b_j is a constant. In this case, power removal is proportional to the temperature. From Eq. B.22, the derivatives of $T_j(t)$ are by inspection

$$T_j^{(k+1)}(t) = (Q_j \, B/c_j) \, N^{(k)}(t) - (b_j/c_j) \, T_j^{(k)}(t) \tag{B.23}$$

$$j=1, \ldots, J$$

for $1 \leq k \leq K$.

The third option available for the heat balance equations is

$$c_j \, dT_j(t)/dt = Q_j \, BN(t) - \sum_{n=1}^{J} h_c^{jn}\left[T_j(t) - T_n(t)\right]$$

$$- \sum_{n=1}^{J} h_r^{jn}\left[T_j^4(t) - T_n^4(t)\right] \qquad j=1, \ldots, J \tag{B.24}$$

which represents heat transfer between the various regions of the reactor by conduction, convection, radiation, or any combination of these. The coefficients h_c^{jn} are the conductive and/or convective heat transfer coefficients from region j to region n, while the coefficients h_r^{jn} are the corresponding radiative heat transfer coefficients. Note that Eq. B.24 is nonlinear because of the $T^4(t)$ terms.

By setting $y_j(t) = T_j^4(t)$ in Eq. B.24, the general expression for the k+1th derivative is seen to be

$$T_j^{(k+1)}(t) = (Q_j B/c_j)\, N^{(k)}(t) - \sum_{n=1}^{J} h_c^{jn}\left[T_j^{(k)}(t) - T_n^{(k)}(t)\right]$$

$$- \sum_{n=1}^{J} h_r^{jn}\left[y_j^{(k)}(t) - y_n^{(k)}(t)\right] \qquad j=1,\ldots,J \qquad (B.25)$$

for $0 \le k \le K$. If $\chi_j(t) = T_j^2(t)$ so that $y_j(t) = \chi_j(t)\,\chi_j(t)$, it can be established by mathematical induction that

$$y_j^{(k)}(t) = \chi_j(t)\,\chi_j^{(k)}(t) + \frac{k}{1!}\,\chi_j^{(1)}\,\chi_j^{(k-1)} + \frac{k(k-1)}{2!}\,\chi_j^{(2)}\,\chi_j^{(k-2)}$$

$$+ \ldots + \chi_j^{(k)}\,\chi_j$$

$$= \chi_j(t)\,\chi_j^{(k)}(t) + \sum_{n=1}^{k} B_n^k\,\chi_j^{(n)}(t)\,\chi_j^{(k-n)}(t) \qquad (B.26)$$

where

$$\chi_j^{(k)}(t) = T_j(t) \ T_j^{(k)}(t) + \frac{k}{1!} \ T_j^{(1)} \ T_j^{(k-1)} + \frac{k(k-1)}{2!} \ T_j^{(2)} \ T_j^{(k-2)}$$

$$+ \ldots + T_j^{(k)} \ T_j$$

$$= T_j(t) \ T_j^{(k)}(t) + \sum_{n=1}^{k} B_n^k \ T_j^{(n)}(t) \ T_j^{(k-n)}(t) \qquad (B.27)$$

for $0 \le k \le K$ and where B_n^k are the binomial coefficients defined in Eq. B.5.

Consider the reactor kinetics equations based on the prompt neutron lifetime. These equations are (Eqs. 3.43, Section 3.4)

$$N^{(1)}(t) = \ell^{-1}[(1 - \beta) \ \delta k(t) - \beta] \ N(t) + \sum_{i=1}^{I} \lambda_i \ C_i(t) + S(t) \qquad (B.28)$$

$$C_i^{(1)}(t) = \ell^{-1} \ \beta_i [\delta k(t) + 1] \ N(t) - \lambda_i \ C_i(t) \qquad i=1, \ldots, I \qquad (B.29)$$

$$\delta k(t) = I(t) + F(t). \qquad (B.30)$$

Equation B.28 is of exactly the same form as Eq. B.1 with $\rho(t)$ and Λ replaced, respectively, by $(1 - \beta) \ \delta k(t)$ and ℓ. Thus, the k+1th derivative of $N(t)$ can be obtained by referring to Eq. B.6:

$$N^{(k+1)}(t) = \ell^{-1}[(1 - \beta) \ \delta k(t) - \beta] \ N^{(k)}(t) + \ell^{-1}(1 - \beta) \sum_{n=1}^{k} B_n^k \ \delta k^{(n)}(t) \ N^{(k-n)}(t)$$

$$+ \sum_{i=1}^{I} \lambda_i \ C_i^{(k)}(t) + S^{(k)}(t). \qquad (B.31)$$

The excess multiplication factor $\delta k(t)$, like the reactivity $\rho(t)$ is expressed as the sum of a forcing function $I(t)$ and a feedback function $F(t)$. Thus, the derivatives of $\delta k(t)$ can be obtained in the same manner as for $\rho(t)$.

Equation B.29 is basically different from its corresponding Eq. B.2 in that Eq. B.2 is linear, while Eq. B.29 is nonlinear because of the term $\delta k(t) \ N(t)$. The higher derivatives of $C_i(t)$ in Eq. B.29 can be established in the same manner as for other nonlinear differential equations considered previously. The results are

$$C_i^{(k+1)}(t) = \ell^{-1} \beta_i [\delta k(t) + 1] N^{(k)}(t) + \ell^{-1} \beta_i \sum_{n=1}^{k} B_n^k \ \delta k^{(n)}(t) \ N^{(k-n)}(t)$$

$$- \lambda_i \ C_i^{(k)}(t) \qquad\qquad i=1, \ldots, I \qquad\qquad (B.32)$$

for $1 \leq k \leq K$.

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APPENDIX C

ANALYTIC SOLUTION FOR CASE 3

The problem defined by the set of equations

$$\frac{dN(t)}{dt} = \frac{\rho(t)}{\Lambda} N(t) \tag{C.1}$$

$$\rho(t) = a + b[T(t) - T(0)] \tag{C.2}$$

$$\frac{dT(t)}{dt} = H\, N(t) \tag{C.3}$$

is one of the few nonlinear cases for which analytic solutions exist. In Section 3.1.3, comparisons were made between approximate results using continuous analytic continuation and the analytic solutions of Eqs. C.1 through C.3. In this appendix, the analytic solution to these equations is displayed.

Differentiation of Eq. C.2 and the use of Eq. C.3 in the results yields

$$\frac{d\rho(t)}{dt} = bH\, N(t). \tag{C.4}$$

Substitution of N(t) from Eq. C.1 into Eq. C.4 gives

$$\rho\, d\rho = bH\, \Lambda\, dN. \tag{C.5}$$

Equation C.5 can be integrated directly to give

$$\frac{1}{2}\left[\rho^2(t) - a^2\right] = bH\Lambda\left[N(t) - N(0)\right], \tag{C.6}$$

where use was made of the fact that $\rho(0) = a$ (Eq. C.2). Equation C.6 can be solved for $\rho(t)$ to yield

$$\rho(t) = \left\{ a^2 + 2bH\Lambda[N(t) - N(0)] \right\}^{\frac{1}{2}}. \tag{C.7}$$

Substitution into Eq. C.1 of $N(t)$ and $\rho(t)$ from Eqs. C.3 and C.2, respectively, gives

$$dN = \frac{1}{H\Lambda} \left\{ a + b[T - T(0)] \right\} dT, \tag{C.8}$$

which upon integration yields

$$N(t) = N(0) + \frac{1}{H\Lambda} \left\{ a[T(t) - T(0)] + \frac{b}{2}[T(t) - T(0)]^2 \right\}. \tag{C.9}$$

Solve for $N(t)$ in Eq. C.3 and substitute the results into Eq. C.9 to obtain

$$\frac{dT(t)}{dt} = HN(0) + \frac{a}{\Lambda}[T(t) - T(0)] + \frac{b}{2\Lambda}[T(t) - T(0)]^2. \tag{C.10}$$

Equation C.10 can be simplified by making the following substitutions:

$$y(t) = T(t) - T(0) \tag{C.11}$$

$$c_1 = b/2\Lambda \qquad c_2 = a/\Lambda \qquad c_3 = HN(0). \tag{C.12}$$

With these substitutions, Eq. C.10 becomes

$$\frac{dy}{dt} = c_1 y^2 + c_2 y + c_3 \tag{C.13}$$

Integration of Eq. C.13 between the limits 0 to t gives

$$\int_0^y \frac{dy}{c_1 y^2 + c_2 y + c_3} = t. \tag{C.14}$$

The integral in Eq. C.14 is in standard form. However, the expression for the integral depends on whether $c_2^2 > 4c_1 c_3$, $c_2^2 = 4c_1 c_3$, or $c_2^2 < 4c_1 c_3$. Only the case in which the temperature coefficient b is negative is of practical interest. For this case, $c_2^2 > 4c_1 c_3$ and the integral is given by

$$(c_2^2 - 4c_1 c_3)^{-\frac{1}{2}} \ln \left[\frac{2c_1 y + c_2 - (c_2^2 - 4c_1 c_3)^{\frac{1}{2}}}{2c_1 y + c_2 + (c_2^2 - 4c_1 c_3)^{\frac{1}{2}}} \right]_0^y = t. \tag{C.15}$$

In order to simplify Eq. C.15, let

$$\omega = (c_2^2 - 4c_1 c_3)^{\frac{1}{2}} = \Lambda^{-1}[a^2 - 2bH \Lambda N(0)]^{\frac{1}{2}} \tag{C.16}$$

$$r_1 = \frac{c_2 + (c_2^2 - 4c_1 c_3)^{\frac{1}{2}}}{2c_1} = \frac{a + \Lambda\omega}{b} \tag{C.17}$$

$$r_2 = \frac{c_2 - (c_2^2 - 4c_1 c_3)^{\frac{1}{2}}}{2c_1} = \frac{a - \Lambda\omega}{b}, \tag{C.18}$$

where use has been made of Eqs. C.12. Substitution of Eqs. C.16 through C.18 into Eq. C.15 yields

$$\ln \left[\left(\frac{y + r_2}{y + r_1} \right) \left(\frac{r_1}{r_2} \right) \right] = \omega t. \tag{C.19}$$

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Equation C.19 can be solved for y, which from Eq. C.11 is equal to $T(t) - T(0)$, to yield

$$T(t) - T(0) = \frac{r_1 r_2 (e^{\omega t} - 1)}{r_1 - r_2 e^{\omega t}} . \tag{C.20}$$

The first derivative of $T(t)$ with respect to t is, from Eq. C.20,

$$\frac{dT(t)}{dt} = \frac{\omega r_1 r_2 (r_1 - r_2) e^{\omega t}}{(r_1 - r_2 e^{\omega t})^2} \tag{C.21}$$

Substitution of Eq. C.21 into Eq. C.3 gives

$$N(t) = \frac{\omega r_1 r_2 (r_1 - r_2) e^{\omega t}}{H(r_1 - r_2 e^{\omega t})^2}$$

$$= \left[\frac{\omega r_1 r_2}{H(r_1 - r_2)} \right] \left(\frac{r_1 - r_2}{r_1 - r_2 e^{\omega t}} \right)^2 e^{\omega t} . \tag{C.22}$$

From Eqs. C.17 and C.18, it is seen that

$$r_1 r_2 = \frac{a^2 - \Lambda^2 \omega^2}{b^2} \tag{C.23}$$

$$r_1 - r_2 = \frac{2 \Lambda \omega}{b} . \tag{C.24}$$

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Substitution of Eq. C.16 into Eq. C.23 gives

$$r_1 r_2 = \frac{2H \Lambda N(0)}{b} \ .$$
(C.25)

With the help of Eqs. C.24 and C.25, it can be seen that

$$\frac{\omega r_1 r_2}{H(r_1 - r_2)} = N(0).$$
(C.26)

Equation C.26 can be substituted into Eq. C.22 to give

$$N(t) = N(0) \left(\frac{r_1 - r_2}{r_1 - r_2 \ e^{\omega t}} \right)^2 e^{\omega t} \ .$$
(C.27)

The solution for $\rho(t)$ can be obtained by combining Eqs. C.2 and C.20; by inspection, the result is

$$\rho(t) = a + \frac{b \ r_1 r_2 (e^{\omega t} - 1)}{r_1 - r_2 \ e^{\omega t}} \ .$$
(C.28)

It can be seen from Eqs. C.1 and C.28 that the instantaneous inverse period, defined by

$$\alpha(t) = \frac{1}{N(t)} \frac{dN(t)}{dt} \ ,$$

is given by

$$\alpha(t) = \frac{a}{\Lambda} + \frac{b \ r_1 r_2 (e^{\omega t} - 1)}{\Lambda (r_1 - r_2 \ e^{\omega t})} \ .$$
(C.29)

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The integrated neutron level is obtained by integration of Eq. C.27. The result is

$$\int_0^t N(t') \, dt' = \frac{N(0)(r_1 - r_2)}{\omega} \left(\frac{e^{\omega t} - 1}{r_1 - r_2 \, e^{\omega t}} \right) . \qquad (C.30)$$

The neutron level reaches a maximum when $dN(t)/dt = 0$. By taking the derivative of Eq. C.27 with respect to time and by setting the result equal to zero, it is found that this occurs at $t = t_m$

where
$$t_m = \frac{1}{\omega} \ln \left(\frac{-r_1}{r_2} \right) . \qquad (C.31)$$

The maximum value of $N(t)$ is obtained by setting $t = t_m$ in Eq. C.28; the result is

$$N(t_m) = N(0) - \frac{a^2}{2\Lambda \, bH} . \qquad (C.32)$$

From Eqs. C.28 and C.20, the reactivity and temperature at $t = t_m$ are given by

$$\rho(t_m) = 0 \qquad (C.33)$$

$$T(t_m) = T(0) - \frac{a}{b} . \qquad (C.34)$$

For $t \to \infty$, Eqs. C.20, C.28, and C.30 yield

$$T(\infty) = T(0) - r_1 \qquad (C.35)$$

$$\rho(\infty) = -\Lambda\omega = -a\left[1 - \frac{2bH \Lambda N(0)}{a^2}\right]^{\frac{1}{2}} \tag{C.36}$$

$$\int_0^\infty N(t') \, dt' = \frac{2\Lambda N(0)}{\Lambda\omega - a} \tag{C.37}$$

If $2bH \Lambda N(0)/a^2 \ll 1$, Eqs. C.35 through C.37 can be approximated by

$$T(\infty) \approx T(0) - \frac{2a}{b}$$

$$\rho(\infty) \approx -a$$

$$\int_0^\infty N(t') \, dt' \approx \frac{-2a}{bH} \quad .$$

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APPENDIX D

ANCON -- A DIGITAL COMPUTER PROGRAM BASED
ON CONTINUOUS ANALYTIC CONTINUATION

A brief description of a digital computer program (ANCON) which solves the reactor kinetics equations by continuous analytic continuation is given in this appendix. The code is programmed in the FORTRAN II, Version 3 language for the IBM-7094 computer and including its associated subroutines, requires a 32K memory.

Because of the occurrence of Λ^{-1} in the reactor kinetics equations, the higher derivatives of $N(t)$ and $C_i(t)$ can become very large when Λ is small (Λ can vary from 10^{-3} sec to 10^{-9} sec). This is a problem only in that there is a limitation on the largest number permissible ($\approx 10^{38}$) in the IBM-7094 computer. Thus, for solution on a digital computer, the equations were rewritten in terms of a dimensionless independent variable τ defined by

$$\tau = \Lambda^{-1} t.$$

In terms of τ, the kinetics equations can be written

$$N^{(1)}(\tau) = [\rho(\tau) - \beta] N(\tau) + \sum_{i=1}^{I} \gamma_i C_i(\tau) + \Lambda S$$

$$C_i^{(1)}(\tau) = \beta_i N(\tau) - \gamma_i C_i(\tau) \qquad\qquad i=1, \ldots, I$$

where $\gamma_i = \Lambda \lambda_i$.

The organization of the computer program is shown in Fig. D.1. The subroutine CLOCK is called by the main program at the beginning of the calculation (after all input data has been read in) and at the end of the calculation (before writing the output). CLOCK records the current
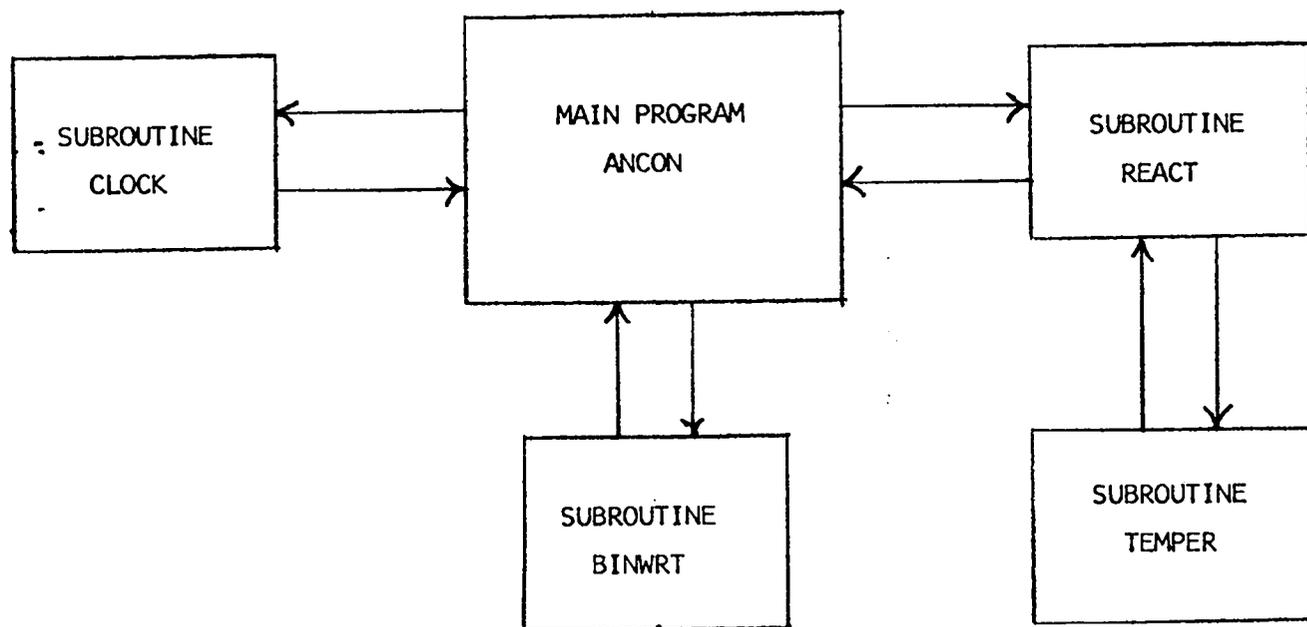
Fig. D.1 Organization of Computer Program

time which is kept to an accuracy of ± 0.01 minute. The subroutine BINWRT is used to punch the output on binary cards. This subroutine is called by the main program at the end of the calculation, if the punched output option has been selected.

The subroutine REACT is called by the main program several times in each time step. This subroutine computes the derivatives of $\rho(t)$ as they are needed by the main program. REACT in turn calls on the subroutine TEMPER, which computes the derivatives of the heat balance equations when they are required by the subroutine REACT.

The program is written in modular form, so that any type of reactivity variation can be represented by adding more options to the subroutines REACT and TEMPER (or by writing new subroutines) and so that modifications to the main program are not required in the process. For example, the feedback model used for the PAERT I transients (Section 3.3.2) required the writing of a special subroutine REACT, since this feedback model was not available in the standard subroutine REACT.

The input to the program is described in Table II. Cards 1 through 5 and card 9 are read in by the main program ANCON. Cards 6 and 7 are read in by REACT, if the option N4 = 2 is selected. This option is for a reactivity variation of the type

$$\rho(t) = \rho(0) + a_1 t + a_2 t^2 + \sum_{j=1}^{J} \alpha_j [T_j(t) - T_j(0)].$$

Card 8 is read in by TEMPER, if N7 = 2, which is the case of heat removal proportional to the temperature:

$$c_j T_j^{(1)}(t) = Q_j BN(t) - b_j T_j(t) \qquad\qquad j=1, \ldots, J.$$

Other options available for $\rho(t)$ and the heat balance equations were described in Appendix B.

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TABLE II

Input Specification for ANCON Code

Card	Format	Entry	Comments
1	12A6	AL(I), I=1,12	Title card
2	12I6	KMAX	Order of Taylor expansion, $KMAX \leq 15$
		IMAX	Number of delayed neutron groups, $IMAX \leq 10$
		N1	N1=0 for new problem, N1≠0 for problem continuation
		N2	N2≠0 if restart dump desired at end of calculation, otherwise N2=0
		N3	N3=1,2, or 3: specifies type of output desired
		N4	N4=1,2, or 3: specifies type of $\rho(t)$ desired
		NMAX	Maximum number of lines of output allowed, $NMAX \leq 999,999$
		N5	N5≠0 if output is to be punched on binary cards, otherwise N5=0, AL(1) is used for identification on cards
		N6	Number of lines printed per line punched on binary cards, $N6 \geq 1$, not used if N5=0
		N8	N8≠0 if $C_i(0)$ to be read in, N8=0 if $C_i(0)$ to be calculated from equilibrium relation $C_i(0) = \beta_i N(0)/\Lambda \lambda_i$
3	6E12.5	E	Error criterion epsilon
		DEL1	Percent change in N(t) before current values of variables are printed
		TIMEST	Value of t at which calculation is terminated
		ANMAX	Maximum number of time steps allowed
		G	Prompt neutron generation time, sec
		S	Source, neuts/sec
		PWRCON	Conversion factor B between neutron level and power level, MW
		FO	N(0)
4	6E12.5	B(I),DC(I) I=1, IMAX	β_i, λ_i, Omit this card if IMAX=0
5	6E12.5	CO(I), I=1, IMAX	$C_i(0)$, Omit this card if IMAX=0 or N8=0

TABLE II (continued)

Card	Format	Entry	Comments
6	2I6,5E12.5	JMAX	Number of heat balance equations, JMAX$\leq$25
		N7	N7=1,2, or 3: specifies type of heat balance equations desired, N7 not used if JMAX=0
		RHOI	$\rho(0)$
		A1	a_1
		A2	a_2
		TMX1	t at which $a_1 t$ term is terminated
		TMX2	t at which $a_2 t^2$ term is terminated

(Note: Cards 6 and 7 are proper input for case N4=2)

Card	Format	Entry	Comments
7	6E12.5	TC(J), J=1, JMAX	Temperature coefficients ($^\circ C^{-1}$), omit this card if JMAX=0
8	6E12.5	TO(J), HCAP(J), Q(J) P(J), J=1, JMAX	$T_j(0)(^\circ K)$, c_j(MW-sec/$^\circ$C), Q_j, b_j

(Note: Card 8 is proper input for case N7=2)

Card	Format	Entry	Comments
9	6E12.5	RESTART DUMP	Restart dump is input as punched out at end of a previous calculation, include only if N1$\neq$0

Some of the input data is edited by ANCON. For example, zero or negative values for $N(0)$, $C_i(0)$, and heat capacities are not allowed and the order of the Taylor expansion must be within the range $1 \leq K \leq 15$. If these tests are not satisfied, appropriate comments are printed and the problem is not executed.

During the calculation, certain variables are continuously tested by ANCON. One of these is the value of the argument in the time step criterion (Section 2.3.1). If this value becomes of the order of the largest number permissible (10^{38}) on the computer, a comment suggesting that K be reduced is printed and the calculation is terminated. A second variable tested is the number of inner iterations (Section 2.3.2). If this number becomes greater than 10 during any time step, a comment suggesting that ε be reduced is printed and the calculation is terminated. Another variable tested is $N(t)$, which must be nonzero and positive and must lie within the range $10^{-30} \leq N(t) \leq 10^{30}$. If the test is not met, a suitable comment is printed and the calculation is terminated.

A problem is terminated whenever t or the number of time steps exceed, respectively, the input parameters TIMEST and ANMAX. Also, the problem can be terminated at any time by depressing sense switch 6.

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